Twisted Morse Complexes

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Why would we study local coefficients?

"Local coefficients bring an extra level of complication that one tries to avoid whenever possible."

– Hatcher, Algebraic Topology, Section 3.H.

"For example, the only way to extend Poincaré duality with $\mathbb{Z}$ coefficients to nonorientable manifolds is to use local coefficients."

– Hatcher, Algebraic Topology, Section 3.H.

Floer homology and symplectic cohomology

Kronheimer and Mrowka use Floer homology of the Seiberg-Witten monopole equation with local coefficients in their book [**Monopoles and Three-manifolds**, New Mathematical Monographs, vol. 10. Cambridge University Press, Cambridge (2007)].

The proof of Viterbo's Theorem, which asserts that there is an isomorphism between the twisted homology of the free loop space of a closed differentiable manifold and the symplectic cohomology of its cotangent bundle, given by Abouzaid uses homology with local coefficients on spaces of piecewise geodesics [Symplectic cohomology and Viterbo's theorem. **Free Loop Spaces in Geometry and Topology**, pp. 271–485 (2015)].

Singular chains

For any $k \in \mathbb{Z}_+$, the **standard k-simplex** Δ^k is the subspace of $\mathbb{R}^{k+1}$ consisting of $(k+1)$-tuples $(t_0, t_1, \ldots, t_k)$ with $t_i \geq 0$ and $t_0 + t_1 + \cdots + t_k = 1$. A **singular k-simplex** in a topological space X is a continuous map $\sigma : \Delta^k \to X$.

For $k \geq 0$, $C_k(X;\mathbb{Z})$ is the free $\mathbb{Z}$-module with generators the singular k-simplices, i.e. an element

$$\sum_{i \in I} a_i \sigma_i \in C_k(X;\mathbb{Z})$$

is a formal sum, where $a_i \in \mathbb{Z}$, the σ_i are k-simplices, and a_i is non-zero for only a finite number of $i \in I$.

If $A \subseteq X$ is a subspace, the inclusion $i : A \to X$ induces a homomorphism $i_* : C_k(A;\mathbb{Z}) \to C_k(X;\mathbb{Z})$, and

$$C_k(X, A;\mathbb{Z}) \overset{\text{def}}{=} C_k(X;\mathbb{Z})/C_k(A;\mathbb{Z}).$$

Singular homology with integer coefficients

There are **face maps** $F_i^k : \Delta^{k-1} \to \Delta^k$ defined by

$$F_i^k(t_0, \ldots, t_{k-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1}) \subset \Delta^k$$

for $0 \leq i \leq k$, which determine a **singular boundary operator**

$$\partial_k : C_k(X;\mathbb{Z}) \to C_{k-1}(X;\mathbb{Z}),$$

defined on a generator $\sigma \in C_k(X;\mathbb{Z})$ by

$$\partial_k(\sigma) = \sigma \circ F_0^k - \sigma \circ F_1^k + \cdots + (-1)^k \sigma \circ F_k^k.$$

It descends to a boundary operator

$$\overline{\partial}_k : C_k(X, A;\mathbb{Z}) \to C_{k-1}(X, A;\mathbb{Z})$$

that satisfies $\overline{\partial}_k \circ \overline{\partial}_{k+1} = 0$, and hence for all $k \geq 0$ we can define

$$H_k(X, A;\mathbb{Z}) \stackrel{\text{def}}{=} Z_k(X, A;\mathbb{Z})/B_k(X, A;\mathbb{Z}) \stackrel{\text{def}}{=} \text{kernel } \overline{\partial}_k/\text{image } \overline{\partial}_{k+1}.$$

Connecting homomorphisms

For any $A \subseteq X$ there is a **connecting homomorphism**

$$\delta_k : H_k(X,A) \to H_{k-1}(A)$$

for all k which fits into the following exact sequence.

$$\cdots \longrightarrow H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X,A) \xrightarrow{\delta_k} H_{k-1}(A) \longrightarrow \cdots$$

For a triple $A \subseteq B \subseteq X$ the connecting homomorphism and the inclusion $j : (B,\emptyset) \to (B,A)$ induce a **connecting homomorphism**

$$\delta_* = j_* \circ \delta_k : H_k(X,B) \xrightarrow{\delta_k} H_{k-1}(B) \xrightarrow{j_*} H_{k-1}(B,A)$$

that fits into the following exact sequence.

$$\cdots \longrightarrow H_k(B,A) \xrightarrow{i_*} H_k(X,A) \xrightarrow{j_*} H_k(X,B) \xrightarrow{\delta_*} H_{k-1}(B,A) \longrightarrow \cdots$$

CW-complexes

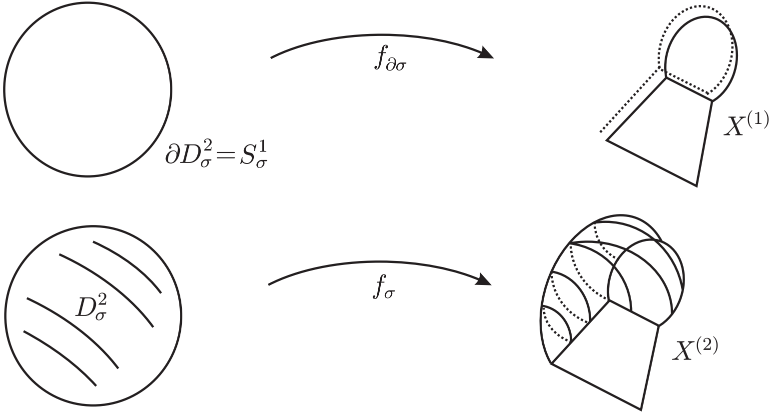
A CW-complex is built step by step by successive operations called **attaching cells**.

Let $D^n \subset \mathbb{R}^n$ be the unit n-disk and $S^{n-1} = \partial D^n$ the unit $(n-1)$-sphere. If $f_\partial : S^{n-1} \to X$ is a continuous map into a topological space X, we denote by

$$X \cup_{f_\partial} D^n$$

the quotient space of the disjoint union $X \amalg D^n$ where $x \in \partial D^n = S^{n-1}$ is identified with $f_\partial(x) \in X$. We say that $X \cup_{f_\partial} D^n$ is obtained from X by **attaching an n-cell** and f_∂ is called the **attaching map**.

Attaching a 2-cell



CW-structures

Definition

A topological space X has a **CW-structure** if there are subspaces $X^{(n)}$ with

$$X^{(0)} \subseteq X^{(1)} \subseteq \cdots \subseteq X = \bigcup_{n \in \mathbb{Z}_+} X^{(n)}$$

such that

- $X^{(0)}$ is a discrete set of points,
- $X^{(n+1)}$ is obtained from $X^{(n)}$ by attaching $(n+1)$-cells for all $n \geq 0$,
- X has the **weak topology**. This means that a subspace of X is open if and only if its intersection with $X^{(n)}$ is open in $X^{(n)}$ for all $n \in \mathbb{Z}_+$.

CW-chains

Lemma

$$H_k(X^{(n)}, X^{(n-1)}; \mathbb{Z}) \approx \begin{cases} \underline{C}_n(X;\mathbb{Z}) & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases}$$

where

$$\underline{C}_n(X;\mathbb{Z}) \approx \bigoplus_\sigma H_n(D^n_\sigma, \partial D^n_\sigma; \mathbb{Z}) \approx \bigoplus_\sigma \mathbb{Z}$$

is the free $\mathbb{Z}$-module generated by the n-cells of X. Moreover, the map

$$\bigoplus_\sigma f_{\sigma *} : \bigoplus_\sigma H_n(D^n_\sigma, \partial D^n_\sigma; \mathbb{Z}) \to H_n(X^{(n)}, X^{(n-1)}; \mathbb{Z})$$

is an isomorphism.

The CW-boundary operator

Define the **CW-boundary operator**

$$\underline{\partial}_n : \underline{C}_n(X;\mathbb{Z}) \to \underline{C}_{n-1}(X;\mathbb{Z})$$

to be the composition

$$\underline{C}_n(X;\mathbb{Z}) \stackrel{\Psi_n}{\to} H_n(X^{(n)}, X^{(n-1)}) \stackrel{\delta_*}{\to} H_{n-1}(X^{(n-1)}, X^{(n-2)}) \stackrel{\Phi_{n-1}}{\to} \underline{C}_{n-1}(X;\mathbb{Z})$$

where

$$\begin{aligned} \Psi_n : \underline{C}_n(X;\mathbb{Z}) &\stackrel{\approx}{\longrightarrow} H_n(X^{(n)}, X^{(n-1)}) \\ \Phi_{n-1} : H_{n-1}(X^{(n-1)}, X^{(n-2)}) &\stackrel{\approx}{\longrightarrow} \underline{C}_{n-1}(X;\mathbb{Z}) \end{aligned}$$

are given by the above lemma, and the map δ_* is the connecting homomorphism of the triple $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$.

The CW-Homology Theorem

Theorem (CW-Homology Theorem)

If X is a CW-complex, then $\underline{\partial}_n : \underline{C}_n(X;\mathbb{Z}) \to \underline{C}_{n-1}(X;\mathbb{Z})$ *satisfies* $\underline{\partial}_{n-1} \circ \underline{\partial}_n = 0$ *and is given by*

$$\underline{\partial}_n(\sigma) = \sum_{\tau} [\sigma : \tau]\tau$$

where $[\sigma : \tau]$ *is the degree of the map* $p_\tau \circ f_{\partial\sigma} : \partial D^n_\sigma \to S^{n-1}_\tau$*. Moreover, there is a natural identification of the homology of the complex* $(\underline{C}_*(X;\mathbb{Z}), \underline{\partial}_*)$ *with the singular homology* $H_*(X;\mathbb{Z})$*.*

In the above theorem, $f_{\partial\sigma} : \partial D^n_\sigma \to X^{(n-1)}$ is the attaching map of the n-cell σ, τ is an $(n-1)$-cell, and p_τ is the composition

$$X^{(n-1)} \to X^{(n-1)}/X^{(n-2)} \to S^{n-1}_\tau.$$

Morse functions

1. The **Hessian** $H_p(f)$ of a smooth function $f : M \to \mathbb{R}$ at a critical point $p \in M$ is a symmetric bilinear map $H_p(f) : T_pM \times T_pM \to \mathbb{R}$ whose matrix in local coordinates $\phi(x) = (x_1, \ldots, x_m)$ is given by

$$M_p(f) = \left(\frac{\partial^2 (f \circ \phi^{-1})}{\partial x_i \partial x_j} \phi(p) \right).$$

2. The dimension of the subspace of T_pM on which $H_p(f)$ is negative definite is called the **index** of p, i.e. the number of negative eigenvalues of $M_p(f)$, and is denote by λ_p.
3. The critical point p is said to be **non-degenerate** if and only if the Hessian $H_p(f)$ is non-degenerate.
4. A **Morse function** $f : M \to \mathbb{R}$ on a smooth manifold M is a smooth function whose critical points are all non-degenerate.

Stable and unstable manifolds

Let $p \in M$ be a critical point of a smooth function $f : M \to \mathbb{R}$ on a smooth Riemannian manifold (M, g) of dimension $m < \infty$, and let $\varphi : \mathbb{R} \times M \to M$ be the 1-parameter family of diffeomorphisms determined by $-\nabla f$. The **stable manifold** of p is

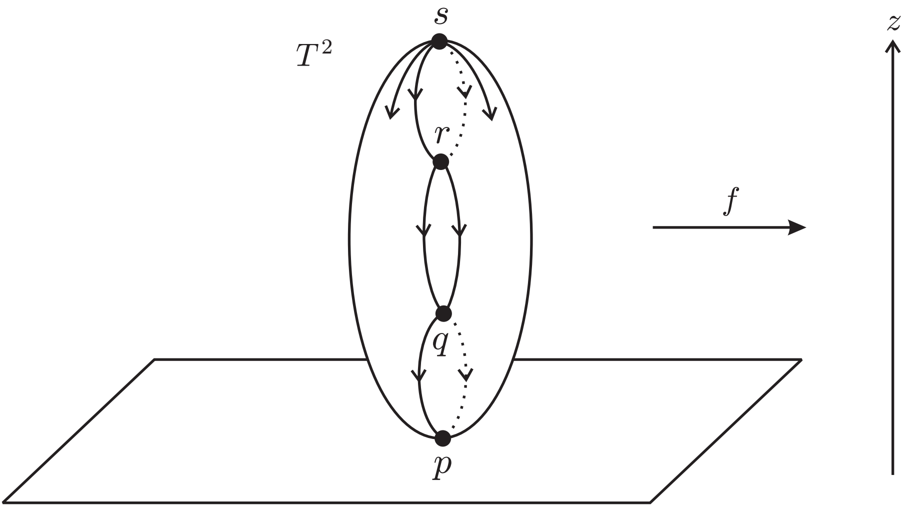
$$W^s(p) = \{x \in M | \lim_{t \to \infty} \varphi_t(x) = p\}$$

and the **unstable manifold** of p is

$$W^u(p) = \{x \in M | \lim_{t \to -\infty} \varphi_t(x) = p\}.$$

The Stable/Unstable Manifold Theorem: If p is a nondegenerate critical point, then the stable manifold $W^s(p)$ is a smoothly embedded open disk of dimension $m - \lambda_p$ and the unstable manifold $W^u(p)$ is a smoothly embedded open disk of dimension λ_p.

Stable and unstable manifolds on T^2



Morse-Smale transversality

A pair (f, g) is called **Morse-Smale** if and only if all the stable and unstable manifolds intersect transversally, i.e. $W^u(q) \pitchfork W^s(p)$ for all $p, q \in Cr(f)$.

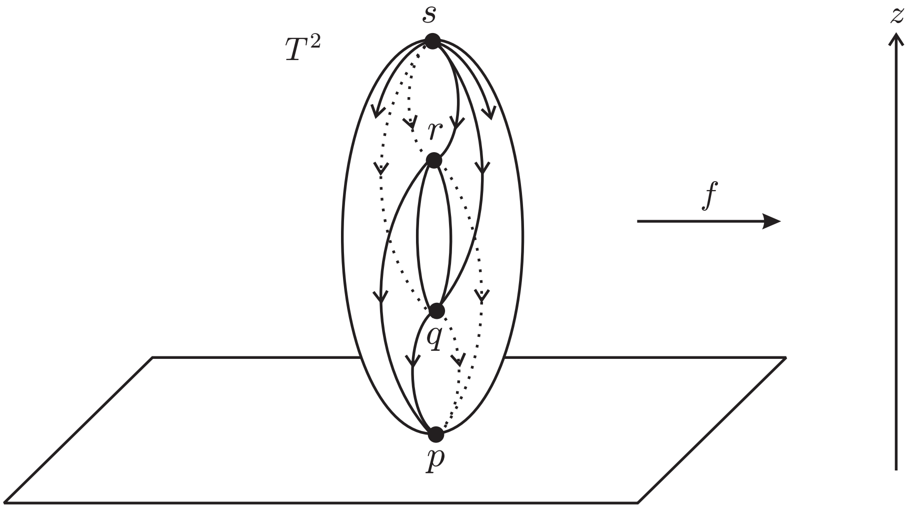
If $W^u(q) \cap W^s(p) \neq \emptyset$, then this condition implies that $W^u(q) \cap W^s(p)$ is a manifold of dimension $\lambda_q - \lambda_p$ and the **moduli space**

$$\mathcal{M}(q,p) = \left(W^u(q) \cap W^s(p)\right)/\mathbb{R}$$

is a manifold of dimension $\lambda_q - \lambda_p - 1$.

Note: The dimension of M **does not affect** the dimension of the moduli space $\mathcal{M}(q,p)$.

A Morse-Smale function on T^2 (tilted)



The Morse-Smale-Witten chain complex

Let $f : M \to \mathbb{R}$ be a Morse-Smale function on a compact smooth Riemannian manifold (M, g) of dimension $m < \infty$, and assume that orientations for the unstable manifolds of f have been chosen. Let $C_k(f)$ be the free abelian group generated by the critical points of index k, and let

$$C_*(f) = \bigoplus_{k=0}^{m} C_k(f).$$

Define a homomorphism $\partial_k : C_k(f) \to C_{k-1}(f)$ by

$$\partial_k(q) \;=\; \sum_{p \in \mathrm{Cr}_{k-1}(f)} n(q,p)p$$

where $n(q,p)$ is the number of gradient flow lines from q to p counted with sign. The pair $(C_*(f), \partial_*)$ is called the **Morse-Smale-Witten chain complex** of f.

Oriented moduli spaces of gradient flow lines

Choosing orientations for all the unstable manifolds $W^u(q)$ determines an orientation on $W(q,p)$ for all $p, q \in Cr(f)$ via

$$T_* W(q,p) \hookrightarrow T_* W^u(q)|_{W(q,p)} \twoheadrightarrow \nu_*(W(q,p), W^u(q))|_{W(q,p)},$$

where the fibers of the normal bundle are isomorphic to $T_p W^u(p)$ via the gradient flow.

The $\lambda_q - \lambda_p - 1$ manifold $\mathcal{M}(q,p) = W(q,p)/\mathbb{R}$ is then oriented by choosing any regular value y between $f(p)$ and $f(q)$, identifying $\mathcal{M}(q,p) = W(q,p) \cap f^{-1}(y)$, and for any $x \in W(q,p) \cap f^{-1}(y)$ declaring B_x to be a positive basis for $T_x\mathcal{M}(q,p)$ if and only if $(-(\nabla f)(x), B_x)$ is a positive basis for $T_x W(q,p)$.

The Morse Homology Theorem

Theorem

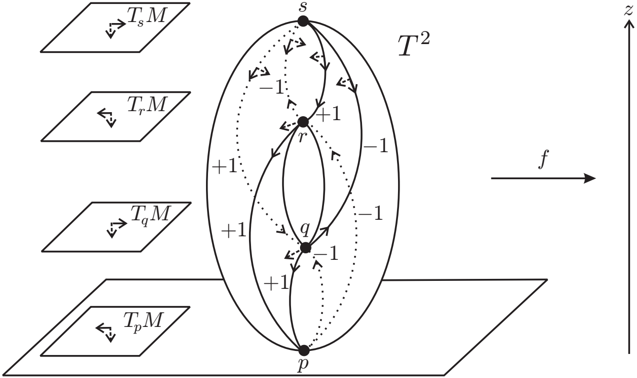
Let $f : M \to \mathbb{R}$ be a Morse function on a smooth manifold M. Suppose that $M^t = \{x \in M | \ f(x) \leq t\}$ is compact for all $t \in \mathbb{R}$. Then M has the homotopy type of a CW-complex X with one cell of dimension k for each critical point of index k.

So, we can use the CW-complex $X \simeq M$ and the CW-Homology Theorem to compute the homology of M, even if (f, g) is not Morse-Smale.

Theorem (Morse Homology Theorem)

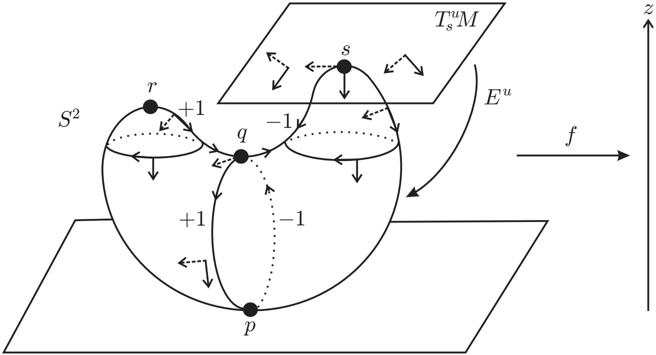
If (f, g) is Morse-Smale, then the pair $(C_*(f), \partial_*)$ is a chain complex, and its homology is isomorphic to the singular homology $H_*(M; \mathbb{Z})$.

The height function on a tilted 2-torus



$$\begin{array}{ccccccc} C_2(f) & \xrightarrow{\partial_2} & C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\ \updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\ < s > & \xrightarrow{\partial_2} & < q, r > & \xrightarrow{\partial_1} & < p > & \longrightarrow & 0 \end{array}$$

The height function on a deformed 2-sphere



$$\begin{array}{ccccccc}
C_2(f) & \xrightarrow{\partial_2} & C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\
\updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\
< r, s > & \xrightarrow{\partial_2} & < q > & \xrightarrow{\partial_1} & < p > & \longrightarrow & 0
\end{array}$$

Unstable manifolds and CW-structures

Theorem (Qin, J. Fixed Point Theory Appl. (2021))

Let $f : M \to \mathbb{R}$ be a Morse-Smale function on a closed, finite dimensional, smooth, Riemannian manifold (M, g).

1. *The unstable manifolds of f determine a CW-structure on M.*
2. *If $q, p \in Cr(f)$ with $\lambda_q - \lambda_p = 1$, then*

$$[\overline{W^u}(q) : \overline{W^u}(p)] = n(q, p).$$

The proof relies on the smooth manifold with corners structure on $\overline{\mathcal{M}(q, p)}$ and topological equivalence.

Similar results were announced or proved earlier for special metrics: Audin and Damian (2014),Burghelea and Haller (2001), Burghelea, Friedlander, and Kappeler (2010), Laudenbach (1992), Qin (2010).

Definition (Bundles of abelian groups)

A **bundle of abelian groups** G over a topological space X associates to every point $x \in X$ an abelian group G_x and to every continuous path $\gamma : [0,1] \to X$ a homomorphism $\gamma_* : G_{\gamma(1)} \to G_{\gamma(0)}$ such that the following conditions are satisfied.

1. If two paths $\gamma_1, \gamma_2 : [0,1] \to X$ from $x \in X$ to $y \in X$ are homotopic rel endpoints, then the homomorphisms from G_y to G_x associated to γ_1 and γ_2 are the same, i.e. $(\gamma_1)_* = (\gamma_2)_*$.
2. If $\gamma : [0,1] \to X$ is constant, then γ_* is the identity.
3. If $\gamma_1, \gamma_2 : [0,1] \to X$ are paths with $\gamma_1(1) = \gamma_2(0)$, then $(\gamma_1\gamma_2)_* = (\gamma_1)_* \circ (\gamma_2)_*$, where $\gamma_1\gamma_2$ denotes the concatenation of γ_1 and γ_2.

Alternately: A bundle of abelian groups G is a functor from the fundamental groupoid of X to the category of abelian groups.

Definition (Isomorphic bundles)

Suppose that G_1 and G_2 are both bundles of abelian groups over a topological space X. If there exists a family of isomorphisms $\Phi : G_1 \to G_2$ such that for every continuous path $\gamma : [0,1] \to X$ the diagram

$$\begin{array}{ccc} (G_1)_{\gamma(1)} & \xrightarrow{\gamma_*^{G_1}} & (G_1)_{\gamma(0)} \\ \downarrow{\scriptstyle \Phi_{\gamma(1)}} & & \downarrow{\scriptstyle \Phi_{\gamma(0)}} \\ (G_2)_{\gamma(1)} & \xrightarrow{\gamma_*^{G_2}} & (G_2)_{\gamma(0)} \end{array}$$

commutes, then G_1 and G_2 are said to be **isomorphic**.

Note: A bundle of abelian groups that is isomorphic to a **constant bundle** ($\gamma_* = id$ for all γ) is called **simple**. A bundle of abelian groups G is simple if and only if for any $x, y \in X$ the homomorphism γ_* is independent of the path γ from x to y.

The η-twisted local coefficient system

Example (The local coefficient system e^{η})

Let $\eta \in \Omega^1_{\mathrm{cl}}(M, \mathbb{R})$ be a closed smooth real valued 1-form on a finite dimensional smooth manifold M. To each point $x \in M$ associate the additive abelian group $\mathbb{R}$, and to each smooth path $\gamma : [0,1] \to M$ associate the homomorphism $\gamma_* : \mathbb{R}_{\gamma(1)} \to \mathbb{R}_{\gamma(0)}$ defined by

$$\gamma_*(s) = e^{\int_1^0 \gamma^*(\eta)} \cdot s \quad \text{for all } s \in \mathbb{R}.$$

This defines a bundle of (additive) $\mathbb{R}$ groups over M, since every continuous path is homotopic rel endpoints to a smooth path.

Lemma

If $\eta_1, \eta_2 \in \Omega^1_{cl}(M, \mathbb{R})$ are in the same de Rham cohomology class, then e^{η_1} is isomorphic to e^{η_2}.

Singular chains with coefficients in G

Let Δ^k denote the standard k-simplex with vertices $e_0, \ldots, e_k$, and let $C_k(X; G)$ be the set of all functions c such that

1. For every singular k-simplex $u : \Delta^k \to X$, $c(u) \in G_{u(e_0)}$ is defined.
2. The set of singular simplices u such that $c(u) \neq 0$ is finite.

Elements of the abelian group $C_k(X; G)$ are called **singular k-chains with coefficients in G**, and every $c \in C_k(X; G)$ can be represented as a finite sum

$$c = \sum_{i=1}^{n} c(u_i) \cdot u_i$$

where $u_1, \ldots, u_n$ are the singular simplices such that $c(u_i) \neq 0$ and $c(u_i) \in G_{u_i(e_0)}$ for all $i = 1, \ldots, n$.

Singular homology with coefficients in G

Definition

The **singular boundary operator with coefficients in** G is defined to be the homomorphism $\partial_k : C_k(X;G) \to C_{k-1}(X;G)$ given on an elementary chain $g \cdot u$ by

$$\partial_k(g \cdot u) = (\gamma_u)_*(g) \cdot u \circ F_0 + \sum_{i=1}^{k} (-1)^i g \cdot u \circ F_i$$

where $(\gamma_u)_* : G_{u(e_0)} \to G_{u(e_1)}$ is the homomorphism associated to the path $\gamma_u(t) = u((1-t)e_1 + t\, e_0)$ from $u(e_1)$ to $u(e_0)$ and $F_i : \Delta^{k-1} \hookrightarrow \Delta^k$ is the inclusion onto the face opposite e_i for all $i = 0, \ldots, k-1$. The pair $(C_*(X;G), \partial_*)$ is a chain complex, and its homology groups $H_*(X;G)$ are called the **singular homology groups of** X **with coefficients in the bundle** G.

Eilenberg's Theorem and equivariant homology

Suppose that (X, x_0) is a connected topological space and G_0 is an abelian group on which $\pi_1(X, x_0)$ acts. There is a chain complex $(G_0 \otimes_{\pi_1} C_*(\widetilde{X}), \bar{\partial}_*)$, where the tensor product is taken over $\pi_1(X, x_0)$ and the boundary operator $\bar{\partial}_*$ is induced from the boundary operator on the singular chains in $\widetilde{X}$. The homology groups of this complex are the **equivariant homology groups** $E_*(\widetilde{X}; G_0)$.

Theorem (Eilenberg)

If G is a bundle of abelian groups in the isomorphism class determined by the action of $\pi_1(X, x_0)$ on G_0, then $H_k(X; G)$ is isomorphic to $E_k(\widetilde{X}; G_0)$ for all k.

Local coefficients on a CW-complex

If G is a local coefficient system on a CW-complex X, the triple $(X^{(k-2)}, X^{(k-1)}, X^{(k)})$ determines a connecting homomorphism

$$H_k(X^{(k)}, X^{(k-1)}; G) \xrightarrow{\delta_k} H_{k-1}(X^{(k-1)}; G)$$

that can be composed with the map

$$H_{k-1}(X^{(k-1)}; G) \xrightarrow{j_*} H_{k-1}(X^{(k-1)}, X^{(k-2)}; G)$$

induced from the inclusion $j : X^{(k-2)} \hookrightarrow X^{(k-1)}$ to give a map

$$H_k(X^{(k)}, X^{(k-1)}; G) \xrightarrow{\tilde{\partial}_k} H_{k-1}(X^{(k-1)}, X^{(k-2)}; G).$$

The above map satisfies $\tilde{\partial}_{k-1} \circ \tilde{\partial}_k = 0$, and the homology groups of the chain complex with boundary operator $\tilde{\partial}_k$ and k^{th}-chain group $H_k(X^{(k)}, X^{(k-1)}; G)$ are isomorphic to the singular homology groups of X with coefficients in the bundle G.

Regular CW-complexes

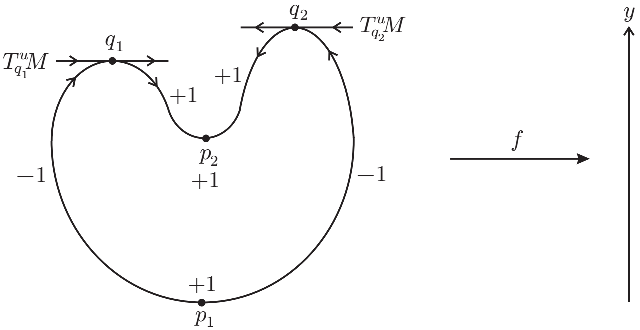
Definition

A CW-complex X is **regular** if every closed k-cell e^k, with $k > 0$, is homeomorphic to Δ^k.

Regular CW-complexes satisfy several properties which are not necessarily satisfied by nonregular CW-complexes. For instance,

1. If $j < k$ and e^j and e^k are cells such that $e^j \cap \dot{e}^k \neq \emptyset$, then $e^j \subset \dot{e}^k$.
2. If e^k and e^{k+2} are cells such that e^k is a face of e^{k+2}, then there are exactly two $(k+1)$-cells e^{k+1} such that e^k is a proper face of e^{k+1} and e^{k+1} is a proper face of e^{k+2}, i.e. $e^k < e^{k+1} < e^{k+2}$.
3. The incidence number $[e^k : e^{k-1}]$ is ± 1 if $e^{k-1} < e^k$ and zero otherwise.

A regular CW-structure on S^1



For each k-cell e^k_σ in a regular CW-complex X choose a basepoint $x(e^k_\sigma)$. This determines an isomorphism

$$\bigoplus_\sigma (f_\sigma)_* : \bigoplus_\sigma H_k(\Delta^k, \dot{\Delta}^k; G_{x(e^k_\sigma)}) \xrightarrow{\approx} H_k(X^{(k)}, X^{(k-1)}; G).$$

The definition of the induced map $(f_\sigma)_*$ requires both a map of spaces $f_\sigma : (\Delta^k, \dot{\Delta}^k) \to (X^{(k)}, X^{(k-1)})$ and a homomorphism $\gamma_* : G_{x(e^k_\sigma)} \to f^*_\sigma(G)$. We take the homomorphism γ_* to be the one defined by restricting the local coefficient system G to the simply connected space e^k_σ. (**This works because X is regular.**) That is, for any point $x \in \Delta^k$ there is a **unique** homotopy class of paths rel endpoints from $f_\sigma(x)$ to $x(e^k_\sigma)$ and hence a well-defined homomorphism $G_{x(e^k_\sigma)} \to G_{f_\sigma(x)}$.

Define

$$CW_k(X; G) \stackrel{\text{def}}{=} \left\{ \sum_\sigma g e^k_\sigma \,\middle|\, g \in G_{x(e^k_\sigma)} \right\} \approx H_k(X^{(k)}, X^{(k-1)}; G)$$

Steenrod's CW-boundary operator

Steenrod's cellular boundary operator with coefficients in G on a regular CW-complex X is defined to be the homomorphism

$$\partial_k : CW_k(X;G) \to CW_{k-1}(X;G)$$

given on an elementary chain ge^k by

$$\partial_k(ge^k) = \sum_{e^{k-1} < e^k} [e^k : e^{k-1}](\gamma_{e^{k-1}e^k})_*(g)e^{k-1},$$

where $(\gamma_{e^{k-1}e^k})_* : G_{x(e^k)} \to G_{x(e^{k-1})}$ denotes the isomorphism determined by any path from $x(e^{k-1})$ to $x(e^k)$ contained in the closure of e^k. We will call the pair $(CW_*(X;G), \partial_*)$ **Steenrod's CW-chain complex with coefficients in the bundle** G.

The Twisted CW-Homology Theorem

Theorem (Twisted CW-Homology Theorem)

If X is a regular CW-complex and G is a bundle of abelian groups over X, then the singular boundary operator with coefficients in G induces Steenrod's cellular boundary operator with coefficients in G. That is, the following diagram commutes.

$$\begin{array}{ccc} CW_k(X;G) & \xrightarrow{\partial_k} & CW_{k-1}(X;G) \\ \updownarrow & & \updownarrow \\ H_k(X^{(k)},X^{(k-1)};G) & \xrightarrow{\tilde{\partial}_k} & H_{k-1}(X^{(k-1)},X^{(k-2)};G) \end{array}$$

Thus, the homology of Steenrod's CW-chain complex $(CW_*(X;G),\partial_*)$ is isomorphic to the singular homology of X with coefficients in the bundle G.

The twisted Morse-Smale-Witten chain complex

Let G be a bundle of abelian groups and (f, g) a Morse-Smale pair on a finite dimensional closed smooth manifold M. Fix orientations on the unstable manifolds, and for all $k = 0, \ldots, m$ define

$$C_k(f;G) \stackrel{def}{=} \left\{ \sum_{q \in Cr_k(f)} gq \,\middle|\, g \in G_q \right\} \approx \bigoplus_{q \in Cr_k(f)} G_q,$$

and $\partial_k^G : C_k(f;G) \to C_{k-1}(f;G)$ by

$$\partial_k^G(gq) = \sum_{p \in Cr_{k-1}(f)} \sum_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu)\gamma_*^\nu(g)p,$$

where $\gamma^\nu : [0,1] \to M$ is any continuous path from p to q whose image coincides with the image of $\nu \in \mathcal{M}(q,p)$ and $\epsilon(\nu) = \pm 1$ is the sign determined by the orientation on $\mathcal{M}(q,p)$.

The η-twisted Morse-Smale-Witten chain complex

Let $\eta \in \Omega^1_{\mathrm{cl}}(M, \mathbb{R})$ be a closed 1-form and (f, g) a Morse-Smale pair on a finite dimensional closed smooth manifold M. We have

$$C_k(f; e^\eta) \approx C_k(f) \otimes \mathbb{R},$$

where $C_k(f)$ is the free abelian group generated by the critical points q of index k. Fixing orientations on the unstable manifolds of (f, g), the homomorphism $\partial_k^\eta : C_k(f) \otimes \mathbb{R} \to C_{k-1}(f) \otimes \mathbb{R}$ is given on a critical point $q \in Cr_k(f)$ by

$$\partial_k^\eta(q) = \sum_{p \in Cr_{k-1}(f)} \sum_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu) \exp\left(\int_{\overline{\mathbb{R}}} \gamma_\nu^*(\eta) \right) p,$$

where $\gamma_\nu : \overline{\mathbb{R}} \to M$ is any gradient flow line from q to p parameterizing $\nu \in \mathcal{M}(q, p)$ and $\epsilon(\nu) = \pm 1$ is the sign determined by the orientation on $\mathcal{M}(q, p)$.

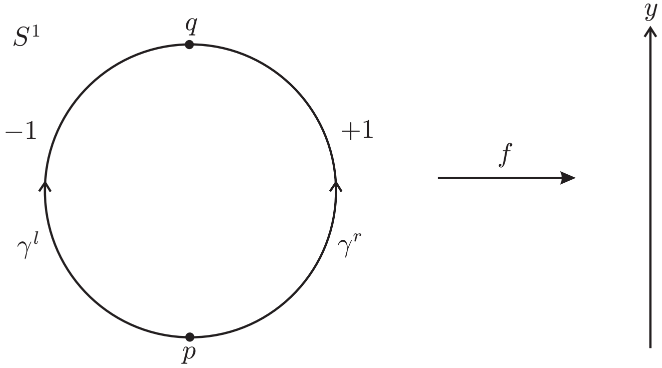
The constant bundle $G = \mathbb{Z}$

If $G = \mathbb{Z}$ is constant and $g \in \mathbb{Z}$, then

$$\begin{aligned}
\partial_k^G(gq) &= \sum_{p \in Cr_{k-1}(f)} \sum_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu)\gamma_*^\nu(g)p \\
&= \sum_{p \in Cr_{k-1}(f)} \sum_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu)\, gp \\
&= \sum_{p \in Cr_{k-1}(f)} g \left(\sum_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu) \right) p \\
&= g \sum_{p \in Cr_{k-1}(f)} n(q,p)p \\
&= g\,\partial_k(q) = \partial_k(gq).
\end{aligned}$$

The height function on a circle

Consider the height function $f : S^1 \to \mathbb{R}$ on the unit circle $S^1 \subset \mathbb{R}^2$ with a critical point q of index 1 and a critical point p of index 0. Orient the unstable manifold of q clockwise and the unstable manifold $W^u(p) = \{p\}$ with $+1$.



The (untwisted) Morse-Smale-Witten chain complex

The (untwisted) Morse-Smale-Witten chain complex of f is

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\ & & \updownarrow \approx & & \updownarrow \approx & & \\ 0 & \longrightarrow & < q > & \xrightarrow{\partial_1} & < p > & \longrightarrow & 0 \end{array}$$

with $\partial_1(q) = 0$ zero since the two gradient flow lines have opposite signs. So,

$$\begin{aligned} H_k(C_*(f), \partial_*) &\approx \mathbb{Z} \text{ if } k = 0, 1 \\ H_k(C_*(f), \partial_*) &\approx 0 \text{ otherwise.} \end{aligned}$$

The η-twisted Morse-Smale-Witten chain complex

Let η be a closed 1-form on S^1 and e^η its associated flat $\mathbb{R}$-bundle. The η-twisted Morse-Smale-Witten boundary operator is given by

$$\partial_1^\eta(q) = \left(\exp\left(\int_1^0 (\gamma^r)^*(\eta)\right) - \exp\left(\int_1^0 (\gamma^l)^*(\eta)\right)\right) p.$$

If η is exact, then $H_*((C_*(f;e^\eta),\partial_*^\eta)) = H_*(S^1;\mathbb{R})$. However, if η is not exact, then

$$\int_1^0 (\gamma^r)^*(\eta) \neq \int_1^0 (\gamma^l)^*(\eta),$$

and $H_k((C_*(f;e^\eta),\partial_*^\eta)) \approx 0$ for all k.

The Twisted Morse Homology Theorem

Theorem (Twisted Morse Homology Theorem)

Let $f : M \to \mathbb{R}$ be a smooth Morse-Smale function on a closed finite dimensional smooth Riemannian manifold (M, g), and let G be a bundle of abelian groups over M. The homology of the Morse-Smale-Witten chain complex with coefficients in G is isomorphic to the singular homology of M with coefficients in G. That is,

$$H_k((C_*(f; G), \partial_*^G)) \approx H_k(M; G)$$

for all $k = 0, \ldots, m$.

Proved by comparing with Steenrod's twisted CW-complex for regular CW-complexes.

Proof outline (invariance)

Theorem (Invariance Theorem)

Let (M, g) be a closed finite dimensional smooth Riemannian manifold, and let G be a bundle of abelian groups over M. Then the homology of the twisted Morse-Smale-Witten chain complex $(C_*(f;G), \partial_*^G)$ is independent of the Morse-Smale pair (f, g) and depends only on the isomorphism class of the bundle of abelian groups G.

Proved in Chapter 3 using standard continuation arguments from Floer theory. The proof relies on the smooth manifolds with corners structure on $\overline{\mathcal{M}}(q,p)$.

Proof outline (triangulations and unstable manifolds)

Theorem (Banyaga, H-, Spaeth)

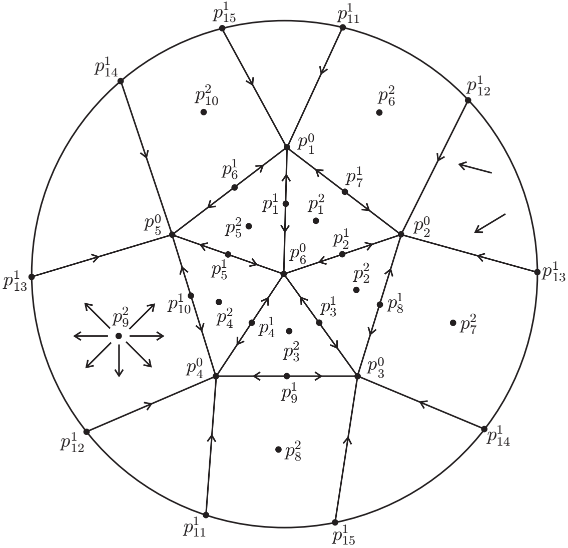
On any closed finite dimensional smooth manifold M there exists a smooth Morse-Smale pair (f, g) such that the unstable manifolds coincide with a smooth triangulation of M. Hence, the unstable manifolds of (f, g) determine a regular CW-structure on M. Moreover, the Riemannian metric g can be chosen such that around every critical point of f there is a Morse chart that is an isometry respect to the standard Euclidean metric on $\mathbb{R}^m$.

Proved in Section 4.4. The fundamental identity

$$\#\mathcal{M}(q, p) = [e_q^k : e_p^{k-1}]$$

follows easily for the function constructed in the above theorem, and hence the Morse-Smale-Witten boundary operator coincides with Steenrod's CW-boundary operator.

A minimal triangulation of $\mathbb{R}P^2$ with ten 2-simplices



Lichnerowicz cohomology (Chapter 5)

For any k-form $\xi \in \Omega^k(M, \mathbb{R})$ define $d_\eta \xi = d\xi + \eta \wedge \xi$. It is easy to verify that $d_\eta \circ d_\eta = 0$, and hence d_η defines a cochain complex

$$\Omega^0(M, \mathbb{R}) \xrightarrow{d_\eta} \Omega^1(M, \mathbb{R}) \xrightarrow{d_\eta} \Omega^2(M, \mathbb{R}) \xrightarrow{d_\eta} \cdots$$

called the **Lichnerowicz cochain complex**.

Theorem (η-Twisted Morse de Rham Theorem)

Let $f : M \to \mathbb{R}$ be a smooth Morse-Smale function on a closed finite dimensional smooth Riemannian manifold (M, g). For any $\eta \in \Omega^1_{cl}(M, \mathbb{R})$, the η-twisted Morse cohomology groups are isomorphic to the Lichnerowicz cohomology groups defined by $-\eta$, i.e.

$$H_k((C^*(f; e^\eta), \delta^\eta_*)) \approx H^k_{-\eta}(M)$$

for all $k = 0, \ldots, m$.

Locally conformal symplectic manifolds

Theorem

Let (M, Ω) be a closed, smooth, finite dimensional LCS manifold with Lee form $\eta \in \Omega^1_{cl}(M, \mathbb{R})$, i.e. $d\Omega = -\eta \wedge \Omega$. Then the η-twisted Morse homology groups $H_*((C_*(f) \otimes \mathbb{R}, \partial^\eta_*))$ and the η-twisted Morse cohomology groups $H_*((C^*(f; e^\eta), \delta^\eta_*))$ are invariants of the conformal class of Ω.

Proved in Chapter 5.

Parallel 1-forms (Section 6.1)

Theorem (Parallel 1-Form Obstruction)

Let $f : M \to \mathbb{R}$ be a smooth Morse-Smale function on a closed finite dimensional smooth Riemannian manifold M, and assume there exists a nonzero closed 1-form η on M such that $H_k((C^*(f;e^\eta),\delta_*^\eta)) \neq 0$ for some k. Then for any nonzero closed 1-form ζ on M such that $[\zeta] = [\eta] \in H^1(M;\mathbb{R})$ the 1-form ζ is not parallel with respect to any Riemannian metric on M.

Proved in Section 6.1 using a result of León, López, Marrero, and Padrón (2003).

H-spaces (Section 6.2)

An H-space is a topological space X together with a continuous map $m : X \times X \to X$ and an element $e \in X$ such that $m(e, \cdot) : X \to X$ and $m(\cdot, e) : X \to X$ are homotopic to the identity through maps that preserve e.

Theorem (Associative H-space Obstruction)

Let (f, g) be a smooth Morse-Smale pair on a closed path connected finite dimensional smooth manifold M. If there exists a local coefficient system $\mathcal{L}$ of rank one vector spaces on M such that

1. *$\mathcal{L}$ is not simple, i.e. $\mathcal{L}$ is not isomorphic to a constant bundle, and*
2. *$H_k((C_*(f; \mathcal{L}), \partial_*^{\mathcal{L}})) \neq 0$ for some k,*

then M is not an associative H-space.

Proved using a result of Albers, Frauenfelder, and Oancea (2017).

Novikov homology (Section 6.3)

S.P. Novikov noted that a closed 1-form ζ on a differentiable manifold M defines a “multivalued function” S by integrating ζ over paths, and S becomes single valued on an appropriate covering space $\widetilde{M} \to M$.

Problem. To construct an analogue of Morse theory for the multivalued functions S. That is, to find a relationship between the stationary points $dS = 0$ of different index and the topology of the manifold M.

Approaches using the dynamics of a flow

The generalization of the Morse-Smale-Witten chain complex to closed 1-forms that determine integral cohomology classes, i.e. to circle valued Morse functions, was carried out by A. Pajitnov (1995), and the construction of a Morse-Smale-Witten type complex using an arbitrary closed 1-form was given by D. Burghelea and S. Haller (2001) and F. Latour (2011).

These generalizations all define the boundary operator for the "Novikov complex" using the dynamics of a flow on a covering of the manifold determined by the closed 1-form and a Riemannian metric. The homology of the Novikov complex is isomorphic to the singular homology of the manifold with local coefficients in a system of rank one $\mathrm{Nov}(\Gamma)$-modules.

In Section 6.3 we use twisted Morse complexes to compute the Novikov numbers of S^1, T^2, K^2, and a surface of genus two.

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