Lichnerowicz cohomology and twisted Morse cohomology

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The Lichnerowicz cochain complex

A closed 1-form $\eta \in \Omega^1_{cl}(M;\mathbb{R})$ on a finite dimensional smooth manifold M can be used to twist the differential of the de Rham cochain complex as follows. For any p-form $\xi \in \Omega^p(M;\mathbb{R})$ define

$$d_\eta \xi = d\xi + \eta \wedge \xi.$$

It is easy to verify that $d_\eta \circ d_\eta = 0$, and hence d_η defines a cochain complex

$$\Omega^0(M;\mathbb{R}) \xrightarrow{d_\eta} \Omega^1(M;\mathbb{R}) \xrightarrow{d_\eta} \Omega^2(M;\mathbb{R}) \xrightarrow{d_\eta} \cdots$$

called the Lichnerowicz cochain complex. The homology of this complex is called the **Lichnerowicz cohomology** $H^*_\eta(M)$.

Since $\eta \in H^1(M;\mathbb{R})$ is closed, for every $\xi \in H^*(M;\mathbb{R})$ we have

$$\begin{aligned} d_\eta(d_\eta(\xi)) &= d_\eta(d\xi + \eta \wedge \xi) \\ &= d(d\xi + \eta \wedge \xi) + \eta \wedge (d\xi + \eta \wedge \xi) \\ &= d\eta \wedge \xi - \eta \wedge d\xi + \eta \wedge d\xi \\ &= 0. \end{aligned}$$

If $\eta = dh$ is exact, then

$$\begin{aligned} e^{-h} d(e^h \xi) &= e^{-h}(e^h dh \wedge \xi + e^h d\xi) \\ &= dh \wedge \xi + d\xi \\ &= d\xi + \eta \wedge \xi = d_\eta \xi, \end{aligned}$$

which shows that d_η is a generalization of the Witten deformation to closed 1-forms.

Locally conformal symplectic manifolds

Definition

A **locally conformal symplectic (LCS)** form Ω on a finite dimensional smooth manifold M is a smooth nondegenerate 2-form such that there exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M and smooth positive functions $\lambda_i > 0$ on each U_i such that $\lambda_i \Omega|_{U_i}$ is a symplectic form on U_i, i.e. $\lambda_i \Omega|_{U_i}$ is closed.

Proposition

If (M, Ω) is an LCS manifold, then the forms $\{d(\ln \lambda_i)\}_{i \in I}$ fit together to give a smooth closed 1-form η such that $d\Omega = -\eta \wedge \Omega$, and η is uniquely determined by the nondegenerate 2-form Ω. Conversely, if Ω is a nondegenerate 2-form on a smooth manifold M such that $d\Omega = -\eta \wedge \Omega$ for some closed 1-form η, then Ω is LCS.

Conformally equivalent LCS manifolds

Definition

The smooth closed 1-form η such that $d\Omega = -\eta \wedge \Omega$ is called the **Lee form** associated to the LCS 2-form Ω.
Two LCS forms Ω and Ω' on M are said to be **conformally equivalent** if and only if there exists a smooth positive function $h > 0$ such that $\Omega' = h\Omega$.

Proposition

If Ω is an LCS form on a finite dimensional smooth manifold M with associated Lee form η and $\Omega' = h\Omega$ for some smooth positive function $h > 0$, then the Lee form associated to Ω' is $\eta - d(\ln h)$. Thus, the de Rham cohomology class of the Lee form $[\eta] \in H^*_{dR}(M; \mathbb{R})$ is an invariant of the conformal class of Ω.

Invariance of Lichnerowicz cohomology

Theorem

For any finite dimensional smooth manifold M, the Lichnerowicz cohomology groups $H_\eta^*(M)$ depend only on the cohomology class $[\eta] \in H_{dR}^*(M;\mathbb{R})$. In particular, if η is exact then the Lichnerowicz cohomology groups are isomorphic to the de Rham cohomology groups, i.e. $H_\eta^k(M) \approx H_{dR}^k(M;\mathbb{R})$ for all $k = 0, \ldots, m$.

This is a generalization of Proposition 4.4 in [Manuel de León, Belén López, Juan C. Marrero, and Edith Padrón, *On the computation of the Lichnerowicz-Jacobi cohomology*, J. Geom. Phys. **44** (2003), no. 4, 507–522; MR 1943175].

Proof of invariance for Lichnerowicz cohomology

Every smooth exact 1-form df can be written as dh/h for some smooth positive function $h > 0$ by setting $h = e^f$. So, if η and η' are closed 1-forms on M with $[\eta] = [\eta'] \in H^1_{\mathrm{dR}}(M;\mathbb{R})$, then there exists a smooth positive function $h : M \to \mathbb{R}$ such that $\eta' = \eta + dh/h$. Define isomorphisms $\phi, \psi : \Omega^k(M;\mathbb{R}) \to \Omega^k(M;\mathbb{R})$ by $\phi(\xi) = \xi/h$ and $\psi(\xi) = h\xi$ for all $k = 0, \ldots, m$.

$$
\begin{aligned}
d_{\eta'}(\phi(\xi)) &= d\left(\frac{1}{h}\xi\right) + \left(\eta + \frac{dh}{h}\right) \wedge \frac{1}{h}\xi \\
&= -\frac{1}{h^2} dh \wedge \xi + \frac{1}{h} d\xi + \eta \wedge \frac{1}{h}\xi + \frac{1}{h^2} dh \wedge \xi \\
&= \frac{1}{h}\left(d\xi + \eta \wedge \xi\right) \\
&= \phi(d_\eta \xi).
\end{aligned}
$$

Proof continued

Similarly,

$$\begin{aligned} d_\eta(\psi(\xi)) &= d(h\xi) + \eta \wedge h\xi \\ &= dh \wedge \xi + hd\xi + \eta \wedge h\xi \\ &= h\left(d\xi + \eta \wedge \xi + \frac{dh}{h} \wedge \xi\right) \\ &= \psi(d_{\eta'}\xi). \end{aligned}$$

Thus, $\phi : (\Omega^*(M;\mathbb{R}), d_\eta) \to (\Omega^*(M;\mathbb{R}), d_{\eta'})$ and $\psi : (\Omega^*(M;\mathbb{R}), d_{\eta'}) \to (\Omega^*(M;\mathbb{R}), d_\eta)$ are chain equivalences and

$$\phi_* : H_k(\Omega^*(M;\mathbb{R}), d_\eta) \to H_k(\Omega^*(M;\mathbb{R}), d_{\eta'})$$

is an isomorphism for all $k = 0, \ldots, m$ with inverse ψ_*.

□

Morse-Smale functions

Let $f : M \to \mathbb{R}$ be a smooth Morse-Smale function on a closed finite dimensional smooth Riemannian manifold (M, α). Let $Cr(f) = \{p \in M | \, df_p = 0\}$ denote the set of critical points of f, and for any $p \in Cr(f)$ let λ_p denote the index of p. For $p, q \in Cr(f)$ let $W^u(q) \subset M$ be the unstable manifold of q, $W^s(p) \subset M$ the stable manifold of p, and define

$$W(q, p) = W^u(q) \pitchfork W^s(p) \subset M.$$

If this space is nonempty, then one says that q is succeeded by p, i.e. $q \succeq p$. In this case, $W(q, p)$ is a noncompact smooth manifold of dimension $\lambda_q - \lambda_p$ by the Morse-Smale transversality condition.

Oriented moduli spaces of gradient flow lines

Choosing orientations for all the unstable manifolds $W^u(q)$ determines an orientation on $W(q,p)$ for all $p, q \in Cr(f)$ via

$$T_*W(q,p) \hookrightarrow T_*W^u(q)|_{W(q,p)} \twoheadrightarrow \nu_*(W(q,p), W^u(q))|_{W(q,p)},$$

where the fibers of the normal bundle are canonically isomorphic to $T_pW^u(p)$ via the gradient flow.

The $\lambda_q - \lambda_p - 1$ manifold $\mathcal{M}(q,p) = W(q,p)/\mathbb{R}$ is then oriented by choosing any regular value y between $f(p)$ and $f(q)$, identifying $\mathcal{M}(q,p) = W(q,p) \cap f^{-1}(y)$, and for any $x \in W(q,p) \cap f^{-1}(y)$ declaring B_x to be a positive basis for $T_x\mathcal{M}(q,p)$ if and only if $(-(\nabla f)(x), B_x)$ is a positive basis for $T_xW(q,p)$.

Compactified moduli spaces as manifolds with corners

The moduli space $\mathcal{M}(q,p)$ has a compactification $\overline{\mathcal{M}}(q,p)$ consisting of the piecewise gradient flow lines from q to p, which can be given the structure of a smooth manifold with corners [D. Burghelea, L. Friedlander, S. Haller, T. Kappeler, F. Latour, L. Qin].

We orient the (codimension) 1-stratum using the convention that an outward pointing normal vector field followed by a positive basis for a tangent space of $\partial^1\overline{\mathcal{M}}(q,p)$ should be a positive basis for a tangent space of $\overline{\mathcal{M}}(q,p)$.

Path components of unparameterized gradient flow lines

A piecewise gradient flow line from q to p can be identified with its image in M, which is an element of $\mathcal{P}^c(M)$, the space of all nonempty closed subsets of M with the Hausdorff topology. This identification is compatible with the topology of the smooth manifold with corners $\overline{\mathcal{M}}(q,p)$ in the sense that the map that sends an element of $\nu \in \overline{\mathcal{M}}(q,p)$ to its image $Im(\nu)$ is a homeomorphism onto its image $Im(\overline{\mathcal{M}}(q,p))$ in $\mathcal{P}^c(M)$. Write $[(\nu_1,\ldots,\nu_l)] = [\nu]$ to indicate that the image of the piecewise gradient flow line $(\nu_1,\ldots,\nu_l)$

$$Im(\nu_1,\ldots,\nu_l) = Im(\gamma_1,\ldots,\gamma_l) = \bigcup_{j=1}^{l} \gamma_j(\overline{\mathbb{R}}) \in \mathcal{P}^c(M)$$

is in the same path component as $Im(\nu)$ in $Im(\overline{\mathcal{M}}(q,p)) \subset \mathcal{P}^c(M)$.

Lemma (Orientations for relative index 2 moduli spaces)

Let $r, p \in Cr(f)$. If $\nu \in \mathcal{M}(r,p)$, then the closure of $\mathcal{M}(r,p;[\nu])$ in $\overline{\mathcal{M}}(r,p)$ consists of the piecewise gradient flow lines from r to p that are in the same path component as ν. Moreover, when $\lambda_r - \lambda_p = 2$ we have

$$\partial\overline{\mathcal{M}}(r,p;[\nu]) = (-1) \bigcup_{\substack{r \succeq q \succeq p \\ [\nu]=[(\nu_1,\nu_2)]}} \mathcal{M}(r,q;[\nu_1]) \times \mathcal{M}(q,p;[\nu_2])$$

as oriented manifolds. Thus when $\lambda_r - \lambda_p = 2$,

$$\sum_{r \succeq q \succeq p} \sum_{\substack{[\nu]=[(\nu_1,\nu_2)] \\ (\nu_1,\nu_2)\in\mathcal{M}(r,q)\times\mathcal{M}(q,p)}} \epsilon(\nu_1)\epsilon(\nu_2) = 0$$

where $\epsilon(\nu_j) = \pm 1$ is the sign of ν_j for $j = 1, 2$.

Definition (η-twisted Morse-Smale-Witten chain complex)

Let $f : M \to \mathbb{R}$ be a smooth Morse-Smale function on a closed finite dimensional smooth Riemannian manifold (M, α), fix orientations on the unstable manifolds of (f, α), and let $\eta \in \Omega^1_{\mathrm{cl}}(M, \mathbb{R})$. The **$\eta$-twisted Morse-Smale-Witten chain complex** is defined to be the chain complex with chain groups $C_k(f) \otimes \mathbb{R}$, where $C_k(f)$ is the free abelian group generated by the critical points q of index k, and the homomorphism $\partial^\eta_k : C_k(f) \otimes \mathbb{R} \to C_{k-1}(f) \otimes \mathbb{R}$ is defined on a critical point by

$$\partial^\eta_k(q) = \sum_{p \in Cr_{k-1}(f)} \sum_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu) \exp\left(\int_{\overline{\mathbb{R}}} \gamma^*_\nu(\eta) \right) p,$$

where $\gamma_\nu : \overline{\mathbb{R}} \to M$ is any gradient flow line from q to p parameterizing $\nu \in \mathcal{M}(q,p)$ and $\epsilon(\nu) = \pm 1$.

Lemma

The pair $(C_*(f) \otimes \mathbb{R}, \partial_*^\eta)$ is a chain complex, i.e. $(\partial_*^\eta)^2 = 0$.

Proof: Let $r \in Cr(f)$ with $\lambda_r = k+1$ for some $k = 1, \ldots, m-1$, where $m = \dim\, M$. We have $\partial_k^\eta(\partial_{k+1}^\eta(r))$

$$
\begin{aligned}
&= \partial_k^\eta \left(\sum_{q \in Cr_k(f)} \sum_{\nu_1 \in \mathcal{M}(r,q)} \exp\left(\int_{\overline{\mathbb{R}}} \gamma_{\nu_1}^*(\eta) \right) \epsilon(\nu_1) q \right) \\
&= \sum_{q \in Cr_k(f)} \sum_{\nu_1 \in \mathcal{M}(r,q)} \exp\left(\int_{\overline{\mathbb{R}}} \gamma_{\nu_1}^*(\eta) \right) \epsilon(\nu_1) \partial_k^\eta(q) \\
&= \sum_{q \in Cr_k(f)} \sum_{\nu_1 \in \mathcal{M}(r,q)} \exp\left(\int_{\overline{\mathbb{R}}} \gamma_{\nu_1}^*(\eta) \right) \epsilon(\nu_1) \sum_{p \in Cr_{k-1}(f)} \sum_{\nu_2 \in \mathcal{M}(q,p)} \exp\left(\int_{\overline{\mathbb{R}}} \gamma_{\nu_2}^*(\eta) \right) \epsilon(\nu_2) p \\
&= \sum_{p \in Cr_{k-1}(f)} \sum_{q \in Cr_k(f)} \sum_{\nu_1 \in \mathcal{M}(r,q)} \sum_{\nu_2 \in \mathcal{M}(q,p)} \exp\left(\int_{\overline{\mathbb{R}}} \gamma_{\nu_1}^*(\eta) + \int_{\overline{\mathbb{R}}} \gamma_{\nu_2}^*(\eta) \right) \epsilon(\nu_1)\epsilon(\nu_2) p.
\end{aligned}
$$

Proof continued

Now fix $p \in Cr_{k-1}(f)$ and consider the coefficient in front of p.

$$\begin{aligned} \text{coef}(p) &= \sum_{q \in Cr_k(f)} \sum_{\nu_1 \in \mathcal{M}(r,q)} \sum_{\nu_2 \in \mathcal{M}(q,p)} \exp\left(\int_{\overline{\mathbb{R}}} \gamma_{\nu_1}^*(\eta) + \int_{\overline{\mathbb{R}}} \gamma_{\nu_2}^*(\eta) \right) \epsilon(\nu_1)\epsilon(\nu_2) \\ &= \sum_{q \in Cr_k(f)} \sum_{(\nu_1,\nu_2) \in \mathcal{M}(r,q) \times \mathcal{M}(q,p)} \exp\left(\int_{\overline{\mathbb{R}}} \gamma_{(\nu_1,\nu_2)}^*(\eta) \right) \epsilon(\nu_1)\epsilon(\nu_2) \end{aligned}$$

where $\gamma_{(\nu_1,\nu_2)} : \overline{\mathbb{R}} \to M$ is any piecewise smooth curve parameterizing the broken gradient flow line (ν_1, ν_2) from r to p. We now group the terms in the above sum according to the various path components $[\nu]$ of $\overline{\mathcal{M}}(r, p)$ and use the fact that the integral is constant on each such path component.

Proof continued

This gives terms of the form

$$\exp\left(\int_{\overline{\mathbb{R}}} \gamma^*(\eta)\right) \sum_{q \in Cr_k(f)} \sum_{\substack{[\nu]=[(\nu_1,\nu_2)] \\ (\nu_1,\nu_2)\in\mathcal{M}(r,q)\times\mathcal{M}(q,p)}} \epsilon(\nu_1)\epsilon(\nu_2)$$

where $\gamma : \overline{\mathbb{R}} \to M$ is any piecewise smooth curve parameterizing an element of $\overline{\mathcal{M}}(r,p;[\nu])$ from r to p. These terms are zero by the lemma on the boundary of moduli spaces of gradient flow lines between critical points of relative index 2.

$\square$

Invariance of η-twisted Morse homology

Theorem (Banyaga, H-, Spaeth)

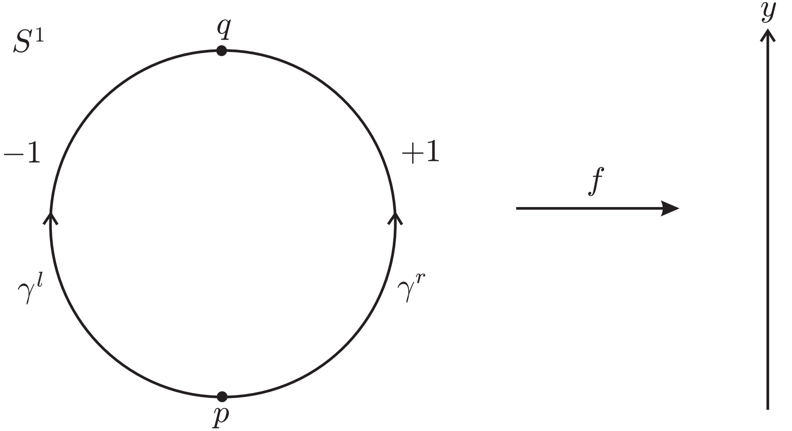
Let $\eta \in \Omega^1_{cl}(M, \mathbb{R})$ be a closed one form on a Riemannian manifold (M, α). Then the homology of the η-twisted Morse-Smale-Witten chain complex $(C_*(f) \otimes \mathbb{R}, \partial^\eta_*)$ is independent of the Morse-Smale pair (f, α) and depends only on the de Rham cohomology class of η.

Corollary

Let (M, Ω) be a closed, smooth, finite dimensional LCS manifold with Lee form $\eta \in \Omega^1_{cl}(M; \mathbb{R})$. Then the η-twisted Morse homology groups $H_*((C_*(f) \otimes \mathbb{R}, \partial^\eta_*))$ are an invariant of the conformal class of Ω.

The height function $f : S^1 \to \mathbb{R}$

The height function on $S^1 \subset \mathbb{R}^2$ has a critical point q of index 1 and a critical point p of index 0. Orient the unstable manifold of q clockwise and the unstable manifold of p as $+1$.



The height function $f : S^1 \to \mathbb{R}$

For a closed 1-form η on S^1 the associated η-twisted Morse-Smale-Witten boundary operator is given by

$$\partial_1^\eta(q) = \left(\exp\left(\int_1^0 (\gamma^r)^*(\eta)\right) - \exp\left(\int_1^0 (\gamma^l)^*(\eta)\right)\right) p.$$

If $\eta = dh$ is exact, then the integral of η along any path from q to p is $h(q) - h(p)$. Hence,

$$\partial_1^\eta(q) = \left(e^{h(q)-h(p)} - e^{h(q)-h(p)}\right) p = 0,$$

and $H_*((C_*(f) \otimes \mathbb{R}), \partial_*^\eta)) = H_*(S^1; \mathbb{R})$.

If η is not exact, then $\int_1^0(\gamma^r)^*(\eta)$ is not equal to $\int_1^0(\gamma^l)^*(\eta)$. In this case $\partial_1^\eta(q) \neq 0$, and $H_k((C_*(f;\mathbb{R}), \partial_*^\eta)) = 0$ for all k.
Explicitly, consider the form,

$$d\theta = \frac{1}{x^2+y^2}(-ydx + xdy)$$

and the parameterization of S^1 given by $\gamma(t) = (\cos t, \sin t)$. Then we have

$$\int_1^0 (\gamma^r)^*(d\theta) = \int_{\pi/2}^{-\pi/2} \sin^2 t + \cos^2 t \; dt = -\pi$$

and

$$\int_1^0 (\gamma^l)^*(d\theta) = \int_{\pi/2}^{3\pi/2} \sin^2 t + \cos^2 t \; dt = \pi.$$

Thus, $\partial_1^\eta(q) = (e^{-\pi} - e^{\pi})p \neq 0$, and $H_k((C_*(f) \otimes \mathbb{R}, \partial_*^\eta)) \approx 0$ for all k.

Definition (Bundles of abelian groups)

A **bundle of abelian groups** G over a topological space X associates to every point $x \in X$ an abelian group G_x and to every continuous path $\gamma : [0,1] \to X$ a homomorphism $\gamma_* : G_{\gamma(1)} \to G_{\gamma(0)}$ such that the following conditions are satisfied.

1. If two paths $\gamma_1, \gamma_2 : [0,1] \to X$ from $x \in X$ to $y \in X$ are homotopic rel endpoints, then the homomorphisms from G_y to G_x associated to γ_1 and γ_2 are the same, i.e. $(\gamma_1)_* = (\gamma_2)_*$.
2. If $\gamma : [0,1] \to X$ is constant, then γ_* is the identity.
3. If $\gamma_1, \gamma_2 : [0,1] \to X$ are paths with $\gamma_1(1) = \gamma_2(0)$, then $(\gamma_1\gamma_2)_* = (\gamma_1)_* \circ (\gamma_2)_*$, where $\gamma_1\gamma_2$ denotes the concatenation of γ_1 and γ_2.

Note: If G is any abelian group and γ_* is the identity map for all paths $\gamma : [0,1] \to X$, then associating $G = G_x$ to every point $x \in X$ determines a **constant** bundle of abelian groups.

Definition (Isomorphic bundles)

Suppose that G_1 and G_2 are both bundles of abelian groups over a topological space X. If there exists a family of isomorphisms $\Phi : G_1 \to G_2$ such that for every continuous path $\gamma : [0,1] \to X$ the diagram

$$\begin{array}{ccc} (G_1)_{\gamma(1)} & \xrightarrow{\gamma_*^{G_1}} & (G_1)_{\gamma(0)} \\ \Big\downarrow \Phi_{\gamma(1)} & & \Big\downarrow \Phi_{\gamma(0)} \\ (G_2)_{\gamma(1)} & \xrightarrow{\gamma_*^{G_2}} & (G_2)_{\gamma(0)} \end{array}$$

commutes, then G_1 and G_2 are said to be **isomorphic**.

Note: A bundle of abelian groups that is isomorphic to a constant bundle is called **simple**. A bundle of abelian groups G is simple if and only if for any $x, y \in X$ the homomorphism γ_* is independent of the path γ from x to y.

The local coefficient system e^η

Definition

Let $\eta \in \Omega^1_{\mathrm{cl}}(M, \mathbb{R})$ be a closed smooth real valued 1-form on a closed finite dimensional smooth manifold M. To each point $x \in M$ associate the additive abelian group $\mathbb{R}$, and to each smooth path $\gamma : [0,1] \to M$ associate the homomorphism $\gamma_* : \mathbb{R}_{\gamma(1)} \to \mathbb{R}_{\gamma(0)}$

$$\gamma_*(s) = e^{\int_1^0 \gamma^*(\eta)} \cdot s \quad \text{for all } s \in \mathbb{R}.$$

Since every continuous path in M is homotopic rel endpoints to a smooth path, Stokes' Theorem shows that this defines a bundle of (additive) $\mathbb{R}$ groups e^η over M, also known as a **flat line bundle**.

Note: The above definition of γ_* extends to paths $\gamma : \overline{\mathbb{R}} \to M$ using any diffeomorphism $\overline{\mathbb{R}} \approx [0,1]$.

$[\eta_1] = [\eta_2]$ implies $e^{\eta_1} \approx e^{\eta_2}$

Claim

If $\eta_1, \eta_2 \in \Omega^1_{cl}(M, \mathbb{R})$ are in the same de Rham cohomology class, then e^{η_1} is isomorphic to e^{η_2}.

Proof: By assumption there exists a smooth function $h : M \to \mathbb{R}$ with $\eta_1 - \eta_2 = dh$. Define a family of isomorphisms $\Phi : e^{\eta_1} \to e^{\eta_2}$ by $\Phi_x(s) = e^{-h(x)} \cdot s$ for all $x \in M$ and $s \in \mathbb{R}$. Then the following diagram commutes for any path $\gamma : [0, 1] \to \mathbb{R}$

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\times e^{\int_1^0 \gamma^*(\eta_1)}} & \mathbb{R} \\ {\scriptstyle \times e^{-h(\gamma(1))}} \downarrow & & \downarrow {\scriptstyle \times e^{-h(\gamma(0))}} \\ \mathbb{R} & \xrightarrow{\times e^{\int_1^0 \gamma^*(\eta_2)}} & \mathbb{R} \end{array}$$

because $e^{\int_1^0 \gamma^*(\eta_1)} = e^{\int_1^0 \gamma^*(\eta_2 + dh)} = e^{\int_1^0 \gamma^*(\eta_2)} e^{h(\gamma(0)) - h(\gamma(1))}$. $\square$

Definition (Twisted Morse-Smale-Witten cochains)

Let $f : M \to \mathbb{R}$ be a smooth Morse-Smale function on a closed smooth Riemannian manifold (M, α) of dimension $m < \infty$. Fix orientations on the unstable manifolds of (f, α), and let G be a bundle of abelian groups over M. For any $k = 0, \ldots, m$, a **Morse-Smale-Witten k-cochain with coefficients in G** is defined to be a function θ that assigns to each critical point $p \in Cr_k(f)$ an element $\theta(p) \in G_p$. The k^{th} **Morse-Smale-Witten cochain group** is the collection of k-cochains, where the group operation is pointwise application of the group operation in G_p. Hence,

$$C^k(f; G) \approx \bigoplus_{p \in Cr_k(f)} G_p.$$

Definition (η-twisted Morse-Smale-Witten cochain complex)

The η-**twisted Morse-Smale-Witten cochain complex** is the chain complex $(C^*(f;e^\eta),\delta_*^\eta)$, where $\delta_k^\eta : C^k(f;e^\eta)\to C^{k+1}(f;e^\eta)$ is defined on a k-cochain $\theta\in C^k(f;e^\eta)$ by

$$(\delta_k^\eta\theta)(q) = \sum_{p\in Cr_k(f)}\sum_{\nu\in\mathcal{M}(q,p)}\epsilon(\nu)\exp\left(\int_{\overline{\mathbb{R}}}(\gamma^\nu)^*(\eta)\right)\theta(p)\in e_q^\eta,$$

for any critical point $q\in Cr_{k+1}(f)$, where $\gamma^\nu:\overline{\mathbb{R}}\to M$ is any continuous path from p to q whose image coincides with the image of $\nu\in\mathcal{M}(q,p)$ and $\epsilon(\nu)=\pm1$ is the sign determined by the orientation on $\mathcal{M}(q,p)$.

The proof that $\delta_{k+1}^\eta\circ\delta_k^\eta=0$ is similar that of $\partial_k^\eta\circ\partial_{k+1}^\eta=0$.

The η-Twisted Morse-Smale-Witten de Rham Theorem

Theorem (Banyaga, H-, Spaeth)

Let $f : M \to \mathbb{R}$ be a smooth Morse-Smale function on a closed finite dimensional smooth Riemannian manifold (M, α). Fix orientations on the unstable manifolds of (f, α) and assume that the unstable manifolds determine a regular CW-structure on M. For any $\eta \in \Omega^1_{cl}(M, \mathbb{R})$, the η-twisted Morse cohomology groups are isomorphic to the Lichnerowicz cohomology groups defined by $-\eta$, i.e.

$$H_k((C^*(f; e^\eta), \delta^\eta_*)) \approx H^k_{-\eta}(M)$$

for all $k = 0, \ldots, m$.

Regular CW-complexes

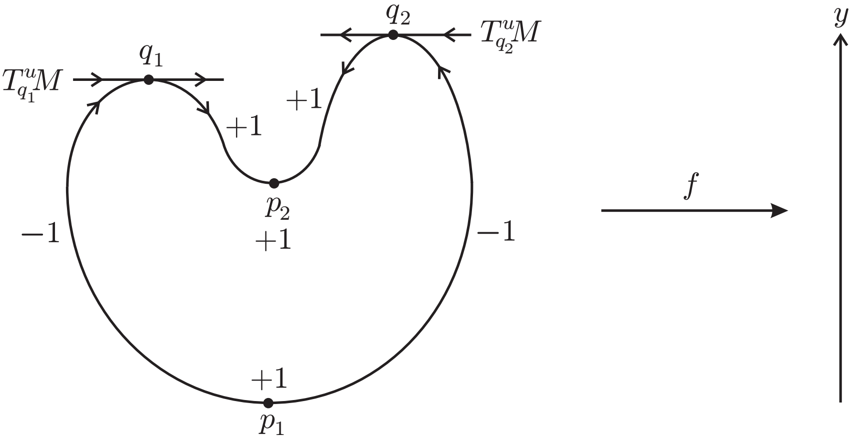
Definition

A CW-complex X is **regular** if every closed k-cell e^k, with $k > 0$, is homeomorphic to Δ^k.

Regular CW-complexes satisfy several properties which are not necessarily satisfied by nonregular CW-complexes. For instance,

1. If $j < k$ and e^j and e^k are cells such that $e^j \cap \dot{e}^k \neq \emptyset$, then $e^j \subset \dot{e}^k$.
2. If e^k and e^{k+2} are cells such that e^k is a face of e^{k+2}, then there are exactly two $(k+1)$-cells e^{k+1} such that e^k is a proper face of e^{k+1} and e^{k+1} is a proper face of e^{k+2}, i.e. $e^k < e^{k+1} < e^{k+2}$.
3. The incidence number $[e^k : e^{k-1}]$ is ± 1 if $e^{k-1} < e^k$ and zero otherwise.

A regular CW-structure on S^1

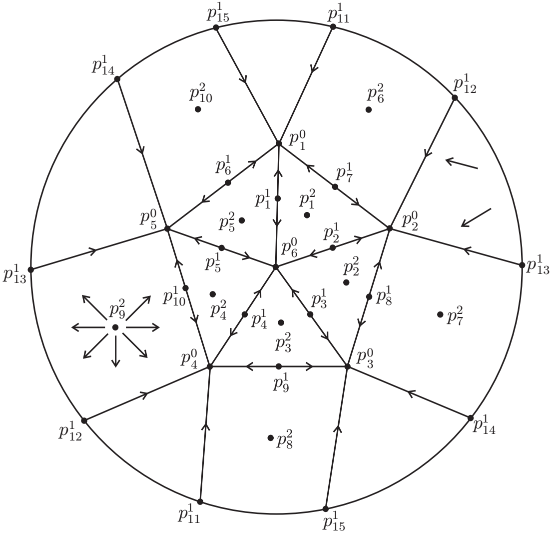


Theorem (Banyaga, H-, Spaeth)

On any closed finite dimensional smooth manifold M there exists a smooth Morse-Smale pair (f, α) such that the unstable manifolds of (f, α) determine a regular CW-structure on M. Moreover, the Riemannian metric α can be chosen so that there are Morse charts of f around every critical point that are isometries with respect to the standard Euclidean metric on $\mathbb{R}^m$.

Proof outline: Pick a triangulation of M fine enough so that every m-simplex is contained in a coordinate chart. Construct a Morse-Smale function with one critical point of index k, for every k-simplex for all $k = 0, \ldots, m$, whose unstable manifolds mimic the triangulation.

Unstable manifolds giving a regular CW-structure on $\mathbb{R}P^2$



Mapping k-forms to Morse-Smale-Witten k-cochains

Fix any $\eta \in \Omega^1_{\text{cl}}(M, \mathbb{R})$ and note that for any $p \in Cr(f)$ the set $U_p \stackrel{\text{def}}{=} \overline{W^u(p)}$ is simply connected since the unstable manifolds of (f, α) determine a regular CW-structure on M. So, $-\eta|_{U_p}$ is exact and $-\eta|_{U_p} = dh/h$ for some smooth positive function $h : U_p \to \mathbb{R}$, if $k = 1, \ldots, m$. For any $\xi \in \Omega^k(M; \mathbb{R})$, where $1 \leq k \leq m$, define

$$\theta_\xi(p) = \frac{1}{h(p)} \int_{U_p} h\xi \in e^\eta_p,$$

and note that this definition is independent of the choice of h, because if $-\eta|_{U_p} = d(\ln \tilde{h}) = d(\ln h)$ then $\tilde{h} = Ch$ for some $C \in \mathbb{R}$. When $k = 0$ define $\theta_\xi(p) = \xi(p)$. This defines a linear map $F : \Omega^k(M; \mathbb{R}) \to C^k(f; e^\eta)$ given by $F(\xi) = \theta_\xi$.

$F : (\Omega^*(M;\mathbb{R}), d_{-\eta}) \to (C^*(f;e^\eta), \delta_*^\eta)$ is a chain map

Pick any $q \in Cr_{k+1}(f)$, let $-\eta|_{U_q} = d(\ln h)$ for some smooth positive function h on $U_q = \overline{W^u(q)} \approx \Delta^{k+1}$, and note that for any $\xi \in \Omega^k(M;\mathbb{R})$ we have

$$d_{-\eta}\xi = d\xi + \frac{dh}{h} \wedge \xi = \frac{1}{h} d(h\xi)$$

on U_q. Moreover, once we fix orientations on the unstable manifolds the signs $\epsilon(\nu) = \pm 1$ satisfy the relation

$$\partial \overline{W^u(q)} = \bigcup_{p \in Cr_k(f)} \bigcup_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu) \overline{W^u(p)}$$

as oriented manifolds.

$$\begin{aligned}
(F \circ d_{-\eta}(\xi))(q) &= \frac{1}{h(q)} \int_{U_q} h d_{-\eta}\xi = \frac{1}{h(q)} \int_{U_q} d(h\xi) = \frac{1}{h(q)} \int_{\partial U_q} h\xi \\
&= \frac{1}{h(q)} \sum_{p \in Cr_k(f)} \sum_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu) \int_{U_p} h\xi \\
&= \frac{1}{h(q)} \sum_{p \in Cr_k(f)} \sum_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu) h(p) \theta_\xi(p) \\
&= \sum_{p \in Cr_k(f)} \sum_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu) e^{\ln h(p) - \ln h(q)} \theta_\xi(p) \\
&= \sum_{p \in Cr_k(f)} \sum_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu) \exp\left(\int_{\overline{\mathbb{R}}} (\gamma^\nu)^*(-d(\ln h)) \right) \theta_\xi(p) \\
&= \sum_{p \in Cr_k(f)} \sum_{\nu \in \mathcal{M}(q,p)} \epsilon(\nu) \exp\left(\int_{\overline{\mathbb{R}}} (\gamma^\nu)^*(\eta) \right) \theta_\xi(p) \\
&= (\delta_k^\eta \circ F(\xi))(q),
\end{aligned}$$

where γ^ν is any parameterization of ν from p to q.

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