

Morse and Morse-Bott Homology

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Betti numbers, Morse theory, and homology

Betti numbers

Morse inequalities

Transversality

Morse homology

Perturbations

Generic perturbations

Applications of the perturbation approach

Morse-Bott inequalities

Cascades

Picture of a 3-cascade

Applications of the cascade approach

Cascades and perturbations

Multicomplexes

Definition and assembly

Multicomplexes and spectral sequences

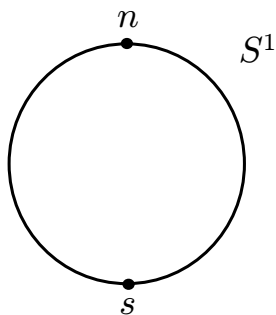
The Morse-Bott-Smale multicomplex

Reference

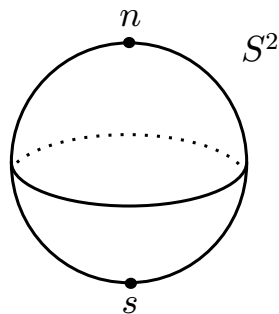
- ▶ David Hurtubise, *Three approaches to Morse-Bott homology*, Afr. Diaspora J. Math. **14** (2012), no. 2, 145–177.

Special volume in honor of Professor Augustin Banyaga on the occasion of his 65th birthday.

Examples for Betti numbers

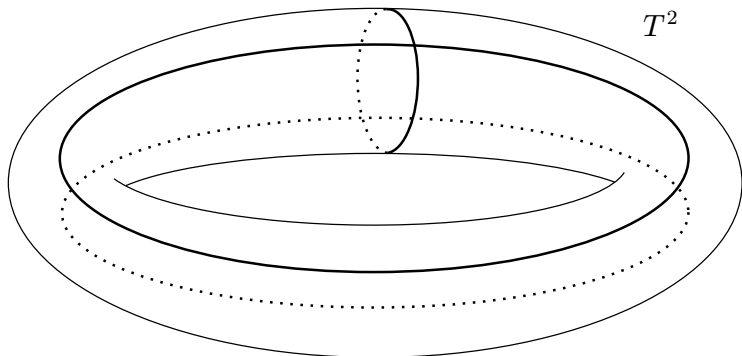


$$\begin{aligned} b_0(S^1) &= 1 \\ b_1(S^1) &= 1 \\ b_2(S^1) &= 0 \end{aligned}$$



$$\begin{aligned} b_0(S^2) &= 1 \\ b_1(S^2) &= 0 \\ b_2(S^2) &= 1 \end{aligned}$$

Examples for Betti numbers

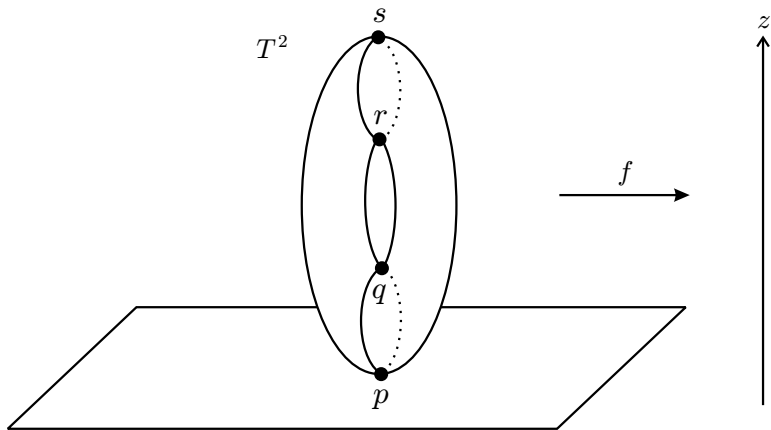


$$b_0(T^2) = 1$$

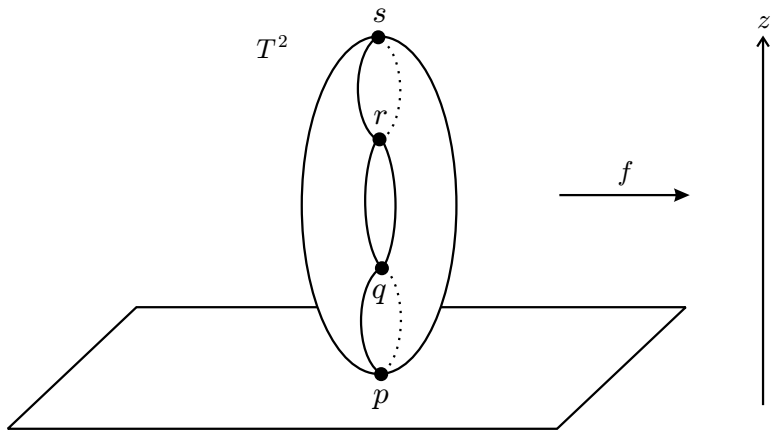
$$b_1(T^2) = 2$$

$$b_2(T^2) = 1$$

Morse functions



Morse functions



Fundamental Idea: There should be a connection between the critical points of a Morse function $f : M \rightarrow \mathbb{R}$ and the Betti numbers of M .

The Morse inequalities

Weak Morse inequalities: $\nu_k(f) \geq b_k(M)$ for all $k = 0, \dots, m$,
 where $\nu_k(f) = \#Cr_k(f)$ is the number of critical points of index k .

Strong Morse inequalities:

$$\begin{array}{rcl} \nu_0 & \geq & b_0 \\ \nu_1 - \nu_0 & \geq & b_1 - b_0 \\ \nu_2 - \nu_1 + \nu_0 & \geq & b_2 - b_1 + b_0 \\ \vdots & \geq & \vdots \\ \nu_{m-1} - \nu_{m-2} + \cdots \pm \nu_0 & \geq & b_{m-1} - b_{m-2} + \cdots \pm b_0 \end{array}$$

$$\nu_m - \nu_{m-1} + \nu_{m-2} - \cdots \pm \nu_0 = b_m - b_{m-1} + b_{m-2} - \cdots \pm b_0$$

Corollary: $\chi(M) = b_0 - b_1 + \cdots \pm b_m = (-1)^m \chi(M)$

The polynomial Morse inequalities

The **Poincaré polynomial** of M is defined to be

$$P_t(M) = \sum_{k=0}^m b_k(M)t^k$$

and the **Morse polynomial** of $f : M \rightarrow \mathbb{R}$ is defined to be

$$M_t(f) = \sum_{k=0}^m \nu_k(f)t^k.$$

Theorem (Polynomial Morse Inequalities): For any Morse function $f : M \rightarrow \mathbb{R}$ on a smooth manifold M we have

$$M_t(f) = P_t(M) + (1+t)R(t)$$

where $R(t)$ is a polynomial with non-negative integer coefficients.

Stable and unstable manifolds

Let $p \in M$ be a critical point of a smooth function $f : M \rightarrow \mathbb{R}$ on a smooth Riemannian manifold M of dimension $m < \infty$, and let $\varphi : \mathbb{R} \times M \rightarrow M$ be the 1-parameter family of diffeomorphisms determined by $-\nabla f$. The **stable manifold** of p is

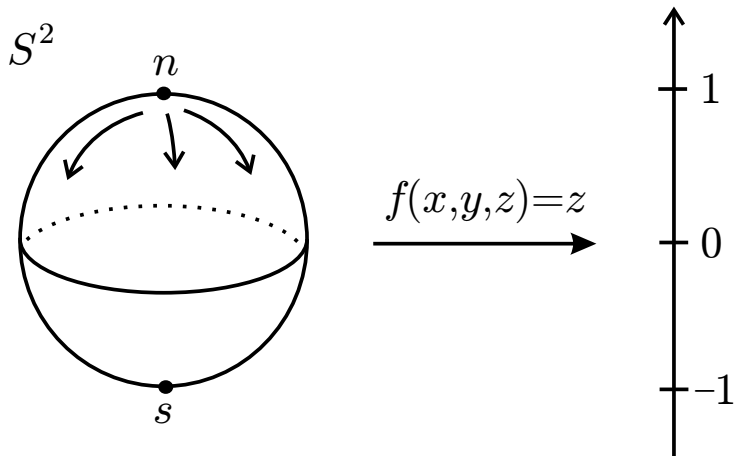
$$W^s(p) = \{x \in M \mid \lim_{t \rightarrow \infty} \varphi_t(x) = p\}$$

and the **unstable manifold** of p is

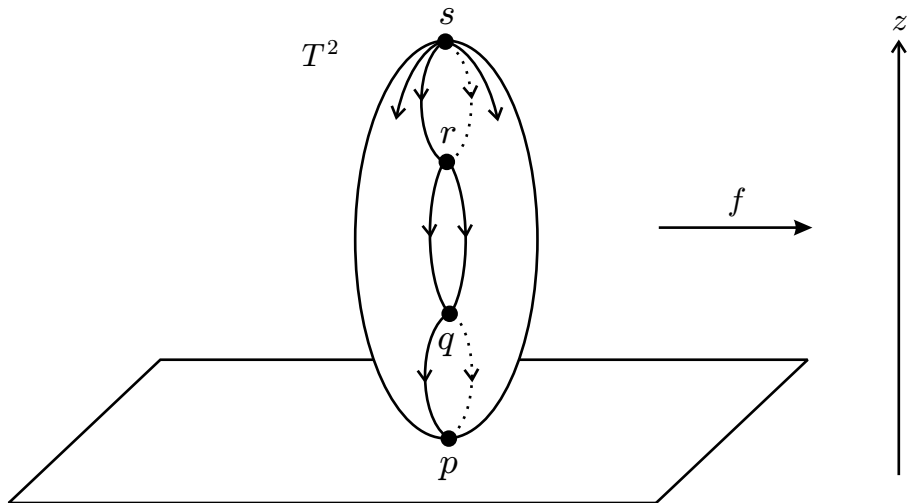
$$W^u(p) = \{x \in M \mid \lim_{t \rightarrow -\infty} \varphi_t(x) = p\}.$$

The Stable/Unstable Manifold Theorem: If p is a nondegenerate critical point, then the stable manifold $W^s(p)$ is a smoothly embedded open disk of dimension $m - \lambda_p$ and the unstable manifold $W^u(p)$ is a smoothly embedded open disk of dimension λ_p .

Examples for stable and unstable manifolds



Examples for stable and unstable manifolds



Morse-Smale transversality

A Morse function $f : M \rightarrow \mathbb{R}$ is called **Morse-Smale** if and only if all its stable and unstable manifolds intersect transversally, i.e. $W^u(q) \pitchfork W^s(p)$ for all $p, q \in Cr(f)$.

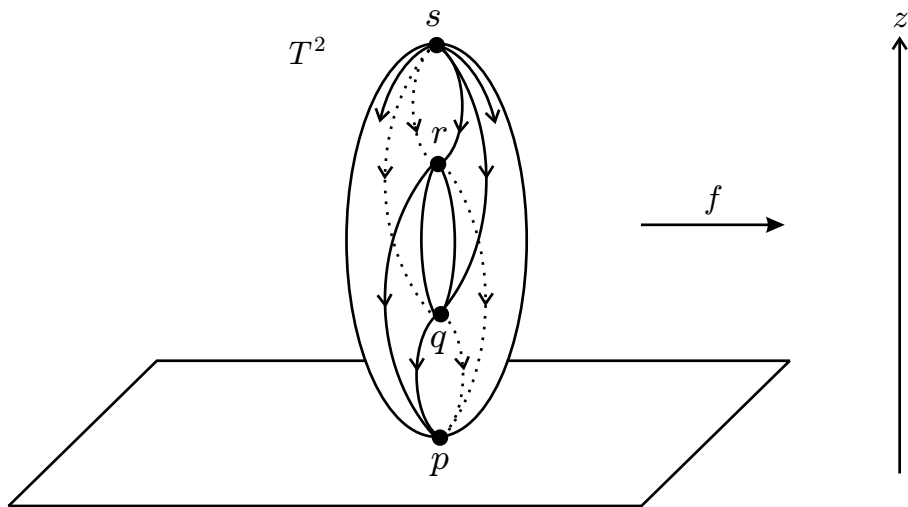
If $W^u(q) \cap W^s(p) \neq \emptyset$, then this condition implies that $W^u(q) \cap W^s(p)$ is a manifold of dimension $\lambda_q - \lambda_p$ and the **moduli space**

$$\mathcal{M}(q, p) = (W^u(q) \cap W^s(p)) / \mathbb{R}$$

is a manifold of dimension $\lambda_q - \lambda_p - 1$.

Note: The dimension of M **does not affect** the dimension of the moduli space $\mathcal{M}(q, p)$.

A Morse-Smale function on a 2-torus



The Morse-Smale-Witten chain complex

Let $f : M \rightarrow \mathbb{R}$ be a Morse-Smale function on a compact smooth Riemannian manifold M of dimension $m < \infty$, and assume that orientations for the unstable manifolds of f have been chosen. Let $C_k(f)$ be the free abelian group generated by the critical points of index k , and let

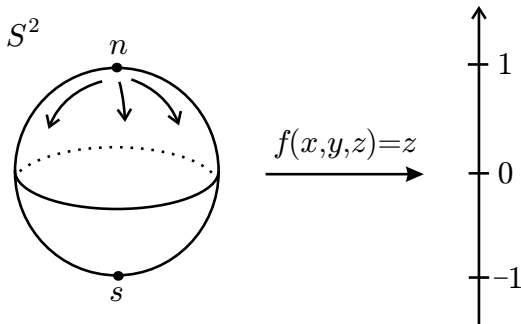
$$C_*(f) = \bigoplus_{k=0}^m C_k(f).$$

Define a homomorphism $\partial_k : C_k(f) \rightarrow C_{k-1}(f)$ by

$$\partial_k(q) = \sum_{p \in \text{Cr}_{k-1}(f)} n(q, p)p$$

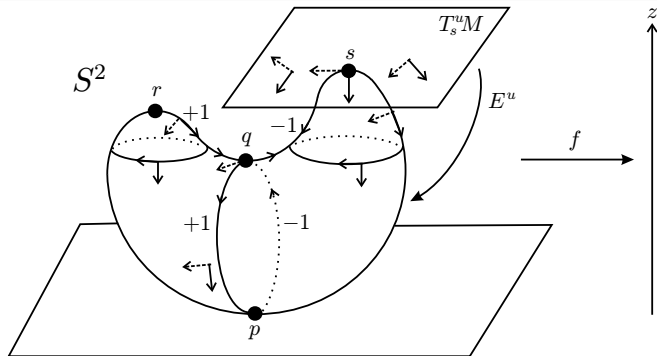
where $n(q, p)$ is the number of gradient flow lines from q to p counted with sign. The pair $(C_*(f), \partial_*)$ is called the **Morse-Smale-Witten chain complex** of f .

The height function on the 2-sphere



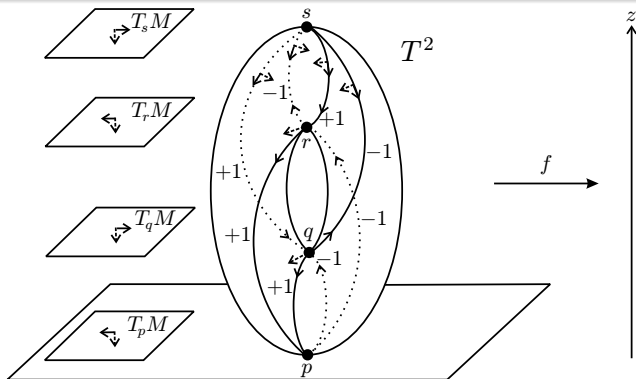
$$\begin{array}{ccccccc}
 C_2(f) & \xrightarrow{\partial_2} & C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\
 \uparrow \approx & & \uparrow \approx & & \uparrow \approx & & \\
 \langle n \rangle & \xrightarrow{\partial_2} & \langle 0 \rangle & \xrightarrow{\partial_1} & \langle s \rangle & \longrightarrow & 0
 \end{array}$$

The height function on a deformed 2-sphere



$$\begin{array}{ccccccc}
 C_2(f) & \xrightarrow{\partial_2} & C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\
 \updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\
 \langle r, s \rangle & \xrightarrow{\partial_2} & \langle q \rangle & \xrightarrow{\partial_1} & \langle p \rangle & \longrightarrow & 0
 \end{array}$$

The height function on a tilted 2-torus

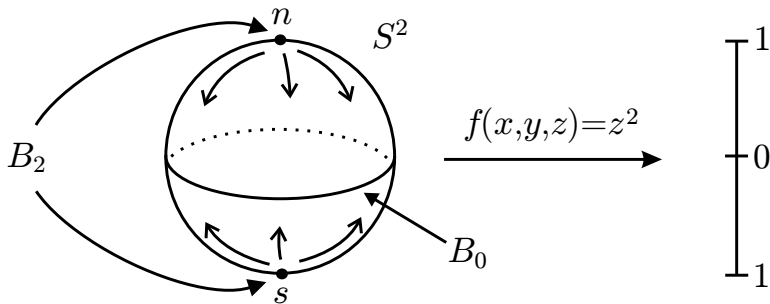


$$\begin{array}{ccccccc}
 C_2(f) & \xrightarrow{\partial_2} & C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\
 \uparrow \approx & & \uparrow \approx & & \uparrow \approx & & \\
 \langle s \rangle & \xrightarrow{\partial_2} & \langle q, r \rangle & \xrightarrow{\partial_1} & \langle p \rangle & \longrightarrow & 0
 \end{array}$$

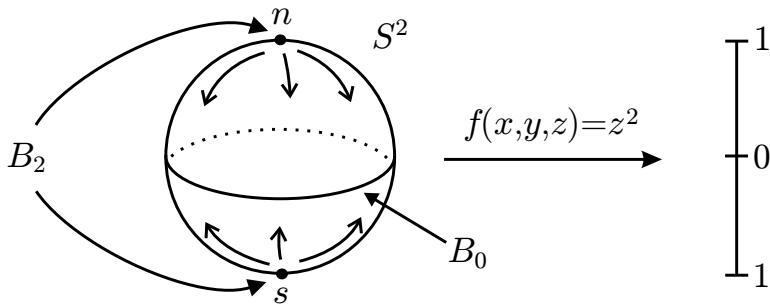
References for Morse homology

- ▶ Augustin Banyaga and David Hurtubise, **Lectures on Morse homology**, Kluwer Texts in the Mathematical Sciences **29**, Kluwer Academic Publishers Group, 2004.
- ▶ John Milnor, **Lectures on the h-cobordism theorem**, Princeton University Press, 1965.
- ▶ Liviu Nicolaescu, **An Invitation to Morse Theory**, *Universitext*, Springer 2007.
- ▶ Matthias Schwarz, **Morse homology**, Progress in Mathematics **111**, Birkhäuser, 1993.
- ▶ Edward Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), no. 4, 661–692.

A Morse-Bott function on the 2-sphere



A Morse-Bott function on the 2-sphere



Can we construct a chain complex for this function? a spectral sequence? a multicomplex?

Generic perturbations

Theorem (Morse 1932)

Let M be a finite dimensional smooth manifold. Given any smooth function $f : M \rightarrow \mathbb{R}$ and any $\varepsilon > 0$, there is a Morse function $g : M \rightarrow \mathbb{R}$ such that $\sup\{|f(x) - g(x)| \mid x \in M\} < \varepsilon$.

Theorem

Let M be a finite dimensional compact smooth manifold. The space of all C^r Morse functions on M is an open dense subspace of $C^r(M, \mathbb{R})$ for any $2 \leq r \leq \infty$ where $C^r(M, \mathbb{R})$ denotes the space of all C^r functions on M with the C^r topology.

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Why not just perturb the Morse-Bott function $f : M \rightarrow \mathbb{R}$ to a Morse function?

Fiber bundles and group actions

If $\pi : E \rightarrow B$ is a smooth fiber bundle with fiber F , and f is a Morse function on B , then $f \circ \pi$ is a Morse-Bott function with critical submanifolds diffeomorphic to F .

$$\begin{array}{ccc}
 F & \longrightarrow & E \\
 & & \downarrow \pi \\
 & & B \longrightarrow \mathbb{R}
 \end{array}$$

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$$\begin{array}{ccc}
 F & \longrightarrow & E \\
 & & \downarrow \pi \\
 & & B \xrightarrow{f} \mathbb{R}
 \end{array}$$

In particular, if G is a Lie group acting on M , then this might be useful for studying equivariant homology.

$$\begin{array}{ccc}
 M & \longrightarrow & EG \times_G M \\
 & & \downarrow \pi \\
 & & BG \xrightarrow{f} \mathbb{R}
 \end{array}$$

Symplectic and Instanton Floer homology

- ▶ Andreas Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988), no. 2, 215–240.
- ▶ Andreas Floer, *Morse theory for Lagrangian intersections*, Journal of Differential Geom **28** (1988), 513–547.
- ▶ Andreas Floer, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989), no. 4, 575–611.

Generalizations: Donaldson polynomials for 4-manifolds with boundary, knot homology groups, comparing the quantum cup product to the pair of pants product.

An explicit perturbation of $f : M \rightarrow \mathbb{R}$

Let T_j be a small tubular neighborhood around each connected component $C_j \subseteq Cr(f)$ for all $j = 1, \dots, l$. Pick a positive Morse function $f_j : C_j \rightarrow \mathbb{R}$ and extend f_j to a function on T_j by making f_j constant in the direction normal to C_j for all $j = 1, \dots, l$.

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Let $\tilde{T}_j \subset T_j$ be a smaller tubular neighborhood of C_j with the same coordinates as T_j , and let ρ_j be a smooth bump function which is constant in the coordinates parallel to C_j , equal to 1 on \tilde{T}_j , equal to 0 outside of T_j , and decreases on $T_j - \tilde{T}_j$ as the coordinates move away from C_j .

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$$h_\varepsilon = f + \varepsilon \left(\sum_{j=1}^l \rho_j f_j \right).$$

Critical points of the perturbed function

If $p \in C_j$ is a critical point of $f_j : C_j \rightarrow \mathbb{R}$ of index λ_p^j , then p is a critical point of h_ε of index

$$\lambda_p^{h_\varepsilon} = \lambda_j + \lambda_p^j$$

where λ_j is the Morse-Bott index of C_j .

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where λ_j is the Morse-Bott index of C_j .

Theorem (Morse-Bott Inequalities)

Let $f : M \rightarrow \mathbb{R}$ be a Morse-Bott function on a finite dimensional oriented compact smooth manifold, and assume that all the critical submanifolds of f are orientable. Then there exists a polynomial $R(t)$ with non-negative integer coefficients such that

$$MB_t(f) = P_t(M) + (1 + t)R(t).$$

(Different orientation assumptions in [Banyaga-H 2009] than the proof using the Thom Isomorphism Theorem.)

The idea behind the Banyaga-H proof

$$\begin{aligned}
 MB_t(f) &= \sum_{j=1}^l P_t(C_j)t^{\lambda_j} \\
 &= \sum_{j=1}^l \left(M_t(f_j) - (1+t)R_j(t) \right) t^{\lambda_j} \\
 &= \sum_{j=1}^l M_t(f_j)t^{\lambda_j} - (1+t) \sum_{j=1}^l R_j(t)t^{\lambda_j} \\
 &= M_t(h) - (1+t) \sum_{j=1}^l R_j(t)t^{\lambda_j} \\
 &= P_t(M) + (1+t)R_h(t) - (1+t) \sum_{j=1}^l R_j(t)t^{\lambda_j}
 \end{aligned}$$

Cascades (Frauenfelder 2003 and Bourgeois 2002)

Let $f : M \rightarrow \mathbb{R}$ be a Morse-Bott function and suppose

$$\text{Cr}(f) = \coprod_{j=1}^l C_j,$$

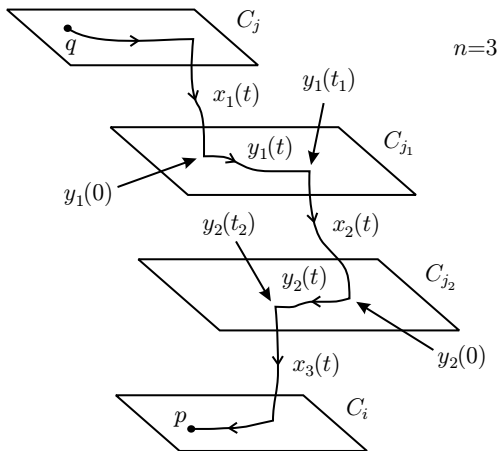
where C_1, \dots, C_l are disjoint connected critical submanifolds of Morse-Bott index $\lambda_1, \dots, \lambda_l$ respectively. Let $f_j : C_j \rightarrow \mathbb{R}$ be a Morse function on the critical submanifold C_j for all $j = 1, \dots, l$.

Definition

If $q \in C_j$ is a critical point of the Morse function $f_j : C_j \rightarrow \mathbb{R}$ for some $j = 1, \dots, l$, then the **total index** of q , denoted λ_q , is defined to be the sum of the Morse-Bott index of C_j and the Morse index of q relative to f_j , i.e.

$$\lambda_q = \lambda_j + \lambda_q^j.$$

A 3-cascade



Definition

Denote the space of flow lines from q to p with n cascades by $W_n^c(q, p)$, and denote the quotient of $W_n^c(q, p)$ by the action of \mathbb{R}^n by $\mathcal{M}_n^c(q, p) = W_n^c(q, p)/\mathbb{R}^n$. The **set of unparameterized flow lines with cascades from q to p** is defined to be

$$\mathcal{M}^c(q, p) = \bigcup_{n \in \mathbb{Z}_+} \mathcal{M}_n^c(q, p)$$

where $\mathcal{M}_0^c(q, p) = W_0^c(q, p)/\mathbb{R}$.

The \mathbb{Z}_2 -cascade chain complex

Define the k^{th} chain group $C_k^c(f)$ to be the free abelian group generated by the critical points of total index k of the Morse-Smale functions f_j for all $j = 1, \dots, l$, and define $n^c(q, p; \mathbb{Z}_2)$ to be the number of flow lines with cascades between a critical point q of total index k and a critical point p of total index $k - 1$ counted mod 2. Let

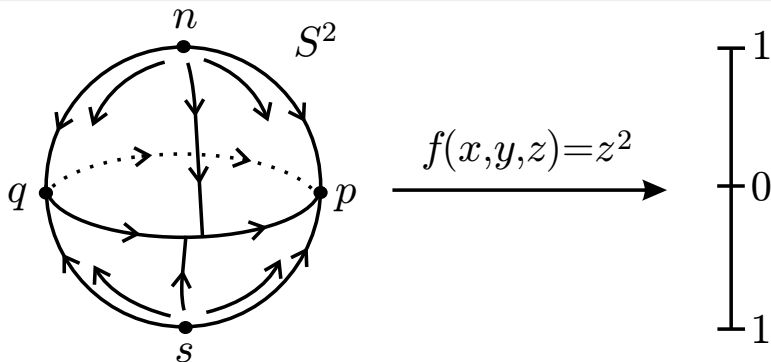
$$C_*^c(f) \otimes \mathbb{Z}_2 = \bigoplus_{k=0}^m C_k^c(f) \otimes \mathbb{Z}_2$$

and define a homomorphism $\partial_k^c : C_k^c(f) \otimes \mathbb{Z}_2 \rightarrow C_{k-1}^c(f) \otimes \mathbb{Z}_2$ by

$$\partial_k^c(q) = \sum_{p \in Cr(f_{k-1})} n^c(q, p; \mathbb{Z}_2) p.$$

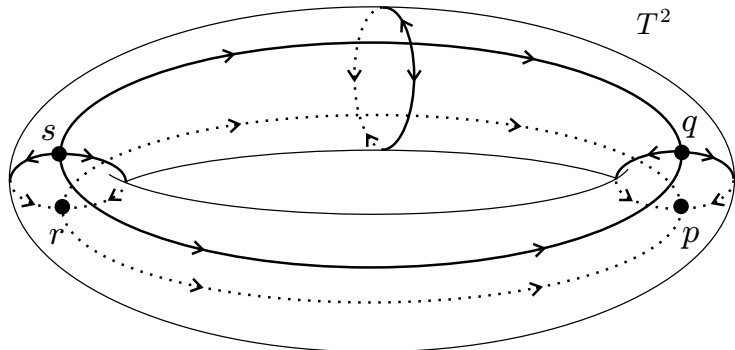
The pair $(C_*^c(f) \otimes \mathbb{Z}_2, \partial_*^c)$ is called the **cascade chain complex** with \mathbb{Z}_2 coefficients.

A cascade chain complex for the 2-sphere



$$\begin{array}{ccccccc}
 C_2^c(f) & \xrightarrow{\partial_2^c} & C_1^c(f) & \xrightarrow{\partial_1^c} & C_0^c(f) & \longrightarrow & 0 \\
 \updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\
 \langle n, s \rangle & \xrightarrow{\partial_2^c} & \langle q \rangle & \xrightarrow{\partial_1^c} & \langle p \rangle & \longrightarrow & 0
 \end{array}$$

A cascade chain complex for the 2-torus



$$\begin{array}{ccccccc}
 C_2^c(f) & \xrightarrow{\partial_2^c} & C_1^c(f) & \xrightarrow{\partial_1^c} & C_0^c(f) & \longrightarrow & 0 \\
 \updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\
 \langle s \rangle & \xrightarrow{\partial_2^c} & \langle q, r \rangle & \xrightarrow{\partial_1^c} & \langle p \rangle & \longrightarrow & 0
 \end{array}$$

The Arnold-Givental conjecture

Let (M, ω) be a $2n$ -dimensional compact symplectic manifold, $L \subset M$ a compact Lagrangian submanifold, and $R \in \text{Diff}(M)$ an antisymplectic involution, i.e. $R^*\omega = -\omega$ and $R^2 = \text{id}$, whose fixed point set is L .

Conjecture. Let H_t be a smooth family of Hamiltonian functions on M for $0 \leq t \leq 1$ and denote by Φ_H the time-1 map of the flow of the Hamiltonian vector field of H_t . If L intersects $\Phi_H(L)$ transversally, then

$$\#(L \cap \Phi_H(L)) \geq \sum_{k=0}^n b_k(L; \mathbb{Z}_2).$$

Proved by Frauenfelder for a class of Lagrangians in Marsden-Weinstein quotients by letting $H \rightarrow 0$ (2004).

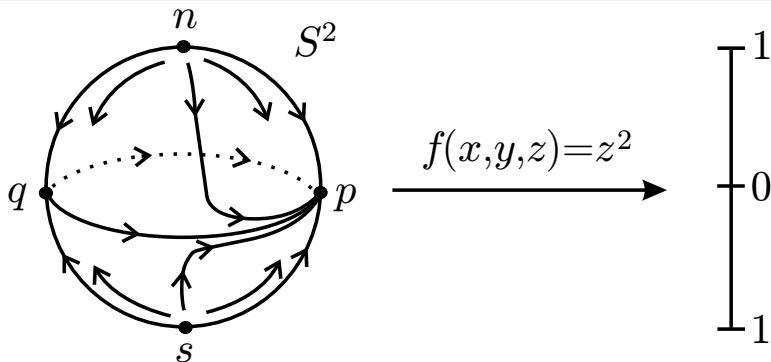
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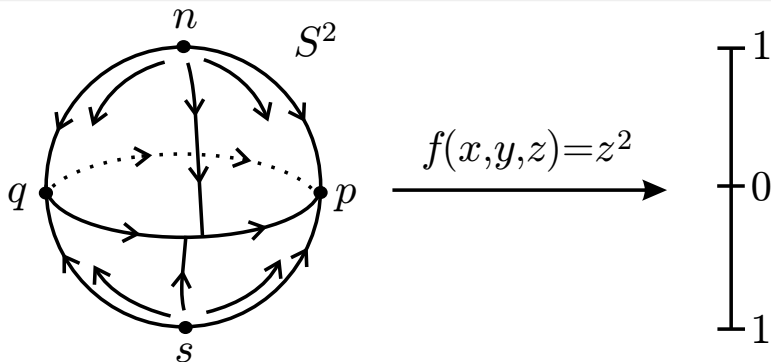
$$h_\varepsilon = f + \varepsilon \left(\sum_{j=1}^l \rho_j f_j \right).$$

A perturbed Morse-Bott function on the 2-sphere



$$\begin{array}{ccccccc}
 C_2(h_\varepsilon) & \xrightarrow{\partial_2} & C_1(h_\varepsilon) & \xrightarrow{\partial_1} & C_0(h_\varepsilon) & \longrightarrow & 0 \\
 \updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\
 \langle n, s \rangle & \xrightarrow{\partial_2} & \langle q \rangle & \xrightarrow{\partial_1} & \langle p \rangle & \longrightarrow & 0
 \end{array}$$

A cascade chain complex for the 2-sphere



$$\begin{array}{ccccccc}
 C_2^c(f) & \xrightarrow{\partial_2^c} & C_1^c(f) & \xrightarrow{\partial_1^c} & C_0^c(f) & \longrightarrow & 0 \\
 \updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\
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 \end{array}$$

Comparing the cascade and Morse chain complexes

For every sufficiently small $\varepsilon > 0$ and $k = 0, \dots, m$ we have

$$C_k^c(f) \approx C_k(h_\varepsilon) = \bigoplus_{\lambda_j + n = k} C_n(f_j).$$

Comparing the cascade and Morse chain complexes

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Is $\mathcal{M}^c(q, p) \approx \mathcal{M}_{h_\varepsilon}(q, p)$ when $\lambda_q - \lambda_p = 1$?

Comparing the cascade and Morse chain complexes

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Is $\mathcal{M}^c(q, p) \approx \mathcal{M}_{h_\varepsilon}(q, p)$ when $\lambda_q - \lambda_p = 1$?

If so, then we can use the orientations on $\mathcal{M}_{h_\varepsilon}(q, p)$ to define the cascade chain complex over \mathbb{Z} so that $\partial_k^c = -\partial_k$ for all $k = 0, \dots, m$, where ∂_k is the Morse-Smale-Witten boundary operator of h_ε . In particular,

$$H_*((C_*(f), \partial_*^c)) \approx H_*(M; \mathbb{Z}).$$

Theorem (Banyaga-H 2013)

Assume that f satisfies the Morse-Bott-Smale transversality condition with respect to the Riemannian metric g on M , $f_k : C_k \rightarrow \mathbb{R}$ satisfies the Morse-Smale transversality condition with respect to the restriction of g to C_k for all $k = 1, \dots, l$, and the unstable and stable manifolds $W_{f_j}^u(q)$ and $W_{f_i}^s(p)$ are transverse to the beginning and endpoint maps.

1. When $n = 0, 1$ the set $\mathcal{M}_n^c(q, p)$ is either empty or a smooth manifold without boundary.
2. For $n > 1$ the set $\mathcal{M}_n^c(q, p)$ is either empty or a smooth manifold with corners.
3. The set $\mathcal{M}^c(q, p)$ is either empty or a smooth manifold without boundary.

In each case the dimension of the manifold is $\lambda_q - \lambda_p - 1$. The above manifolds are orientable when M and C_k are orientable.

Correspondence of moduli spaces

Theorem (Banyaga-H 2013)

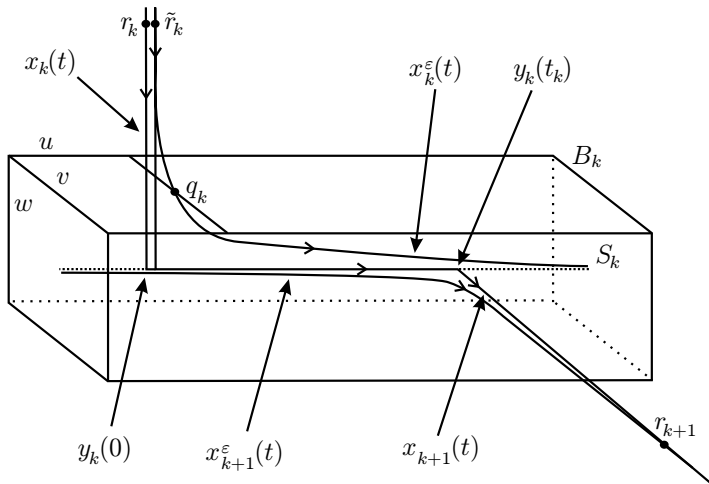
Let $p, q \in \text{Cr}(h_\varepsilon)$ with $\lambda_q - \lambda_p = 1$. For any sufficiently small $\varepsilon > 0$ there is a bijection between unparameterized cascades and unparameterized gradient flow lines of the Morse-Smale function $h_\varepsilon : M \rightarrow \mathbb{R}$ between q and p ,

$$\mathcal{M}^c(q, p) \leftrightarrow \mathcal{M}_{h_\varepsilon}(q, p).$$

Definition

Let $p, q \in \text{Cr}(h_\varepsilon)$ with $\lambda_q - \lambda_p = 1$, define an orientation on the zero dimensional manifold $\mathcal{M}^c(q, p)$ by identifying it with the left hand boundary of $\mathcal{M}_{h_\varepsilon}(q, p) \times [0, \varepsilon]$.

Main idea: The Exchange Lemma



Correspondence of chain complexes

Theorem (Banyaga-H 2011)

For $\varepsilon > 0$ sufficiently small we have $C_k^c(f) = C_k(h_\varepsilon)$ and $\partial_k^c = -\partial_k$ for all $k = 0, \dots, m$, where ∂_k denotes the Morse-Smale-Witten boundary operator determined by the Morse-Smale function h_ε . In particular, $(C_^c(f), \partial_*^c)$ is a chain complex whose homology is isomorphic to the singular homology $H_*(M; \mathbb{Z})$.*

Moral: The cascade chain complex of a Morse-Bott function $f : M \rightarrow \mathbb{R}$ is **the same** as the Morse-Smale-Witten complex of a small perturbation of f .

Multicomplexes

Let R be a principal ideal domain. A first quadrant **multicomplex** X is a bigraded R -module $\{X_{p,q}\}_{p,q \in \mathbb{Z}_+}$ with differentials

$$d_i : X_{p,q} \rightarrow X_{p-i,q+i-1} \quad \text{for all } i = 0, 1, \dots$$

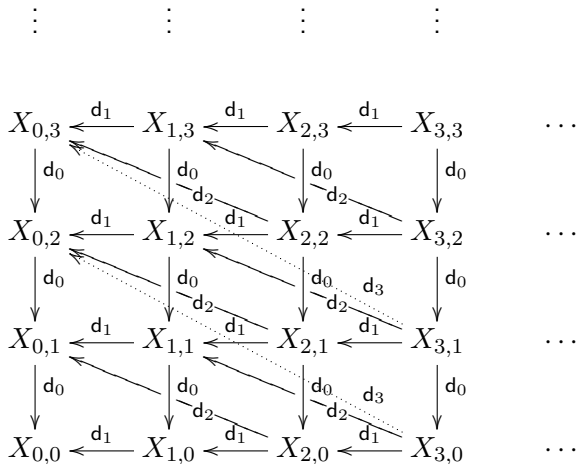
that satisfy

$$\sum_{i+j=n} d_i d_j = 0 \quad \text{for all } n.$$

A first quadrant multicomplex can be **assembled** to form a filtered chain complex $((CX)_*, \partial)$ by summing along the diagonals, i.e.

$$(CX)_n \equiv \bigoplus_{p+q=n} X_{p,q} \quad \text{and} \quad F_s(CX)_n \equiv \bigoplus_{\substack{p+q=n \\ p \leq s}} X_{p,q}$$

and $\partial_n = d_0 \oplus \dots \oplus d_n$ for all $n \in \mathbb{Z}_+$. The above relations then imply that $\partial_n \circ \partial_{n+1} = 0$ and $\partial_n(F_s(CX)_*) \subseteq F_s(CX)_*$.



A bicomplex has two filtrations, but a general multicomplex only has one filtration.

$$\begin{array}{cccccccc}
 \dots & X_{3,0} & \xrightarrow{d_0} & 0 & & & & \\
 & \oplus & \searrow^{d_1} & \oplus & & & & \\
 \dots & X_{2,1} & \xrightarrow{d_0} & X_{2,0} & \xrightarrow{d_0} & 0 & & \\
 & \oplus & \searrow^{d_1} & \oplus & \searrow^{d_1} & \oplus & & \\
 \dots & X_{1,2} & \xrightarrow{d_0} & X_{1,1} & \xrightarrow{d_0} & X_{1,0} & \xrightarrow{d_0} & 0 \\
 & \oplus & \searrow^{d_1} & \oplus & \searrow^{d_1} & \oplus & \searrow^{d_1} & \oplus \\
 \dots & X_{0,3} & \xrightarrow{d_0} & X_{0,2} & \xrightarrow{d_0} & X_{0,1} & \xrightarrow{d_0} & X_{0,0} & \xrightarrow{d_0} & 0 \\
 & \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
 \dots & (CX)_3 & \xrightarrow{\partial_3} & (CX)_2 & \xrightarrow{\partial_2} & (CX)_1 & \xrightarrow{\partial_1} & (CX)_0 & \xrightarrow{\partial_0} & 0
 \end{array}$$

(Note: Dotted lines in the original image connect $X_{3,0}$ to $X_{2,1}$, $X_{2,1}$ to $X_{1,2}$, and $X_{1,2}$ to $X_{0,3}$ via diagonal arrows labeled d_3, d_2, d_1 respectively.)

The bigraded module associated to the filtration

$$F_s(CX)_n \equiv \bigoplus_{\substack{p+q=n \\ p \leq s}} X_{p,q}$$

is

$$G((CX)_*)_{s,t} = F_s(CX)_{s+t} / F_{s-1}(CX)_{s+t} \approx X_{s,t}$$

for all $s, t \in \mathbb{Z}_+$, and the E^1 term of the associated spectral sequence is given by

$$E_{s,t}^1 = Z_{s,t}^1 / (Z_{s-1,t+1}^0 + \partial Z_{s,t+1}^0)$$

where

$$Z_{s,t}^1 = \{c \in F_s(CX)_{s+t} \mid \partial c \in F_{s-1}(CX)_{s+t-1}\}$$

$$Z_{s,t}^0 = \{c \in F_s(CX)_{s+t} \mid \partial c \in F_s(CX)_{s+t-1}\} = F_s(CX)_{s+t}.$$

$E_{s,t}^1$ and d^1 are induced from d_0 and d_1

Theorem

Let $(\{X_{p,q}\}_{p,q \in \mathbb{Z}_+}, \{d_i\}_{i \in \mathbb{Z}_+})$ be a first quadrant multicomplex and $((CX)_*, \partial)$ the associated assembled chain complex. Then the E^1 term of the spectral sequence associated to the filtration of $(CX)_*$ determined by the restriction $p \leq s$ is given by $E_{s,t}^1 \approx H_{s+t}(X_{s,*}, d_0)$ where $(X_{s,*}, d_0)$ denotes the following chain complex.

$$\dots \xrightarrow{d_0} X_{s,3} \xrightarrow{d_0} X_{s,2} \xrightarrow{d_0} X_{s,1} \xrightarrow{d_0} X_{s,0} \xrightarrow{d_0} 0$$

Moreover, the d^1 differential on the E^1 term of the spectral sequence is induced from the homomorphism d_1 in the multicomplex.

However, d^r is not induced from d_r for $r \geq 2$

Consider the following first quadrant double complex

$$\begin{array}{ccccc}
 0 & \xleftarrow{d_1} & 0 & \xleftarrow{d_1} & 0 \\
 \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\
 \langle x_{0,1} \rangle & \xleftarrow{d_1} & \langle x_{1,1} \rangle & \xleftarrow{d_1} & 0 \\
 \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\
 0 & \xleftarrow{d_1} & \langle x_{1,0} \rangle & \xleftarrow{d_1} & \langle x_{2,0} \rangle
 \end{array}$$

where $\langle x_{p,q} \rangle$ denotes the free abelian group generated by $x_{p,q}$, the groups $X_{p,q} = 0$ for $p + q > 2$, and the homomorphisms d_0 and d_1 satisfy the following: $d_0(x_{1,1}) = x_{1,0}$, $d_1(x_{1,1}) = x_{0,1}$, and $d_1(x_{2,0}) = x_{1,0}$. In this case, $d_2 = 0$ but $d^2 \neq 0$

The Morse-Bott-Smale multicomplex

Let $C_p(B_i)$ be the group of “ p -dimensional chains” in the critical submanifolds of index i . Assume that $f : M \rightarrow \mathbb{R}$ is a Morse-Bott-Smale function and the manifold M , the critical submanifolds, and their negative normal bundles are all orientable.

If $\sigma : P \rightarrow B_i$ is a singular C_p -space in $S_p^\infty(B_i)$, then for any $j = 1, \dots, i$ composing the projection map π_2 onto the second component of $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ with the endpoint map $\partial_+ : \overline{\mathcal{M}}(B_i, B_{i-j}) \rightarrow B_{i-j}$ gives a map

$$P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\pi_2} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\partial_+} B_{i-j}.$$

$$\begin{array}{ccccccc}
 \dots & & \vdots & & & & \\
 & & \oplus & & & & \\
 \dots & C_1(B_2) & \xrightarrow{\partial_0} & C_0(B_2) & \xrightarrow{\partial_0} & 0 & \\
 & \searrow^{\partial_1} & & \searrow^{\partial_1} & & & \\
 & \oplus & & \oplus & & \oplus & \\
 \dots & C_2(B_1) & \xrightarrow{\partial_0} & C_1(B_1) & \xrightarrow{\partial_0} & C_0(B_1) & \xrightarrow{\partial_0} & 0 \\
 & \searrow^{\partial_1} & & \searrow^{\partial_1} & & \searrow^{\partial_1} & & \\
 & \oplus & & \oplus & & \oplus & & \oplus \\
 \dots & C_3(B_0) & \xrightarrow{\partial_0} & C_2(B_0) & \xrightarrow{\partial_0} & C_1(B_0) & \xrightarrow{\partial_0} & C_0(B_0) & \xrightarrow{\partial_0} & 0 \\
 & \parallel & & \parallel & & \parallel & & \parallel & & \\
 \dots & C_3(f) & \xrightarrow{\partial} & C_2(f) & \xrightarrow{\partial} & C_1(f) & \xrightarrow{\partial} & C_0(f) & \xrightarrow{\partial} & 0
 \end{array}$$

The Morse-Bott Homology Theorem

Theorem (Banyaga-H 2010)

The homology of the Morse-Bott-Smale multicomplex $(C_(f), \partial)$ is independent of the Morse-Bott-Smale function $f : M \rightarrow \mathbb{R}$.*

Therefore,

$$H_*(C_*(f), \partial) \approx H_*(M; \mathbb{Z}).$$

Note: If f is constant, then $(C_*(f), \partial)$ is the chain complex of singular N -cube chains. If f is Morse-Smale, then $(C_*(f), \partial)$ is the Morse-Smale-Witten chain complex. This gives a new proof of the Morse Homology Theorem.

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The Chern-Simons functional

Let $P \rightarrow N$ be a (trivial) principal $SU(2)$ -bundle over an oriented closed 3-manifold N , and let \mathcal{A} be the space of connections on P . Define $CS : \mathcal{A} \rightarrow \mathbb{R}$ by

$$CS(A) = \frac{1}{4\pi^2} \int_M \text{tr} \left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A \right).$$

The above functional descends to a function $cs : \mathcal{A}/\mathcal{G} \rightarrow \mathbb{R}/\mathbb{Z}$ whose critical points are gauge equivalence classes of flat connections. Extending everything to $P \times \mathbb{R} \rightarrow N \times \mathbb{R}$, the gradient flow equation becomes the instanton equation

$$F + *F = 0,$$

where F denotes the curvature and $*$ is the Hodge star operator.

Instanton homology

Andreas Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988), no. 2, 215–240.

Theorem. When N is a homology 3-sphere the Chern-Simons functional can be perturbed so that it has discrete critical points and defines \mathbb{Z}_8 -graded homology groups $I_*(N)$ analogous to the Morse homology groups.

Generalizations: Donaldson polynomials for 4-manifolds with boundary, knot homology groups

The symplectic action functional

Let (M, ω) be a closed symplectic manifold and $S^1 = \mathbb{R}/\mathbb{Z}$. A time-dependent Hamiltonian $H : M \times S^1 \rightarrow \mathbb{R}$ determines a time-dependent vector field X_H by

$$\omega(X_H(x, t), v) = v(H)(x, t) \text{ for } v \in T_x M.$$

Let $\mathcal{L}(M)$ be the space of free contractible loops on M and

$$\tilde{\mathcal{L}}(M) = \{(x, u) \mid x \in \mathcal{L}(M), u : D^2 \rightarrow M \text{ such that } u(e^{2\pi i t}) = x(t)\} / \sim$$

its universal cover with covering group $\pi_2(M)$. The symplectic action functional $a_H : \tilde{\mathcal{L}}(M) \rightarrow \mathbb{R}$ is defined by

$$a_H((x, u)) = \int_{D^2} u^* \omega + \int_0^1 H(x(t), t) dt.$$

The Arnold conjecture

Andreas Floer, *Symplectic fixed points and holomorphic spheres*,
Comm. Math. Phys. **120** (1989), no. 4, 575–611.

Theorem. Let (P, ω) be a compact symplectic manifold. If I_ω and I_c are proportional, then the fixed point set of every exact diffeomorphism of (P, ω) satisfies the Morse inequalities with respect to any coefficient ring **whenever it is nondegenerate**.

Generalizations: Allowing H to be degenerate (e.g. $H = 0$) leads to critical submanifolds and Morse-Bott homology.

The Yang-Mills gradient flow

Let (Σ, g) be a closed oriented Riemann surface, G a compact Lie group, \mathfrak{g} its Lie algebra, and P a principal G -bundle over Σ . Pick an ad-invariant inner product on \mathfrak{g} , let $\mathcal{A}(P)$ denote the affine space of \mathfrak{g} -valued connection 1-forms on P , and define $\mathcal{YM} : \mathcal{A}(P) \rightarrow \mathbb{R}$ by

$$\mathcal{YM}(A) = \int_{\Sigma} F_A \wedge *F_A$$

where $F_A = dA + \frac{1}{2}[A \wedge A]$ is the curvature of A .

The Yang-Mills function is a Morse-Bott function studied by Atiyah-Bott and by Swoboda (2011) using cascades.

Closed Reeb orbits

Let M be a compact, orientable manifold of dimension $2n - 1$ with contact form α . The **Reeb vector field** R_α associated to the contact form α is characterized by

$$\begin{aligned}d\alpha(R_\alpha, -) &= 0 \\ \alpha(R_\alpha) &= 1.\end{aligned}$$

Closed trajectories of the Reeb vector field are critical points of the action functional $\mathcal{A} : C^\infty(S^1, M) \rightarrow \mathbb{R}$

$$\mathcal{A}(\gamma) = \int_\gamma \alpha.$$

Lemma. For any contact structure ξ on M , there exists a contact form α for ξ such that all closed orbits of R_α are nondegenerate.

Contact homology

Let \mathbf{A} be the graded supercommutative algebra freely generated by the “good” closed Reeb orbits over the graded ring $\mathbb{Q}[H_2(M; \mathbb{Z})/\mathcal{R}]$, i.e. $\gamma_1\gamma_2 = (-1)^{|\gamma_1||\gamma_2|}\gamma_2\gamma_1$.

Theorem. (Eliashberg-Hofer 2000) There is a differential $d : \mathbf{A} \rightarrow \mathbf{A}$ defined by counting J -holomorphic curves in the symplectization $(\mathbb{R} \times M, d(e^t\alpha))$ such that (\mathbf{A}, d) is a **differential graded algebra**. Moreover, $HC_*(M, \xi) \stackrel{\text{def}}{=} H_*(\mathbf{A}, d)$ is an invariant of the contact structure ξ .

Theorem. (Bourgeois 2002) Assume that α is a contact form of **Morse-Bott type** for (M, ξ) and that J is an almost complex structure on the symplectization that is S^1 -invariant along the critical submanifolds N_T . Then there is a chain complex with a boundary operator defined by counting **cascades** whose homology is isomorphic to the contact homology $HC_*(M, \xi)$.

Viterbo's symplectic homology

Definition

A compact symplectic manifold (W, ω) has **contact type** boundary if and only if there exists a vector field X defined in a neighborhood of $M = \partial W$ transverse and pointing outward along M such that $\mathcal{L}_X \omega = \omega$.

In this case, $\lambda = \omega(X, \cdot)|_M$ is a contact form on M , and the symplectic homology of W combines the 1-periodic orbits of a Hamiltonian on W with the Reeb orbits on $M = \partial W$.

Bourgeois and Oancea have defined the cascade chain complex for a time-independent Hamiltonian on W whose 1-periodic orbits are transversally nondegenerate (2009). They have also proved that there is an exact sequence relating the symplectic homology groups of W with the linearized contact homology groups of M (2009).

Compactness

Denote the space of nonempty closed subsets of $M \times \overline{\mathbb{R}}^l$ in the topology determined by the Hausdorff metric by $\mathcal{P}^c(M \times \overline{\mathbb{R}}^l)$, and map a broken flow line with cascades (v_1, \dots, v_n) to its image $\text{Im}(v_1, \dots, v_n) \subset M$ and the time t_j spent flowing along or resting on each critical submanifold C_j for all $j = 1, \dots, l$.

Theorem (Banyaga-H 2013)

The space $\overline{\mathcal{M}}^c(q, p)$ of broken flow lines with cascades from q to p is compact, and there is a continuous embedding

$$\mathcal{M}^c(q, p) \hookrightarrow \overline{\mathcal{M}}^c(q, p) \subset \mathcal{P}^c(M \times \overline{\mathbb{R}}^l).$$

Hence, every sequence of unparameterized flow lines with cascades from q to p has a subsequence that converges to a broken flow line with cascades from q to p .