Morse-Bott Homology

(Using singular N-cube chains)

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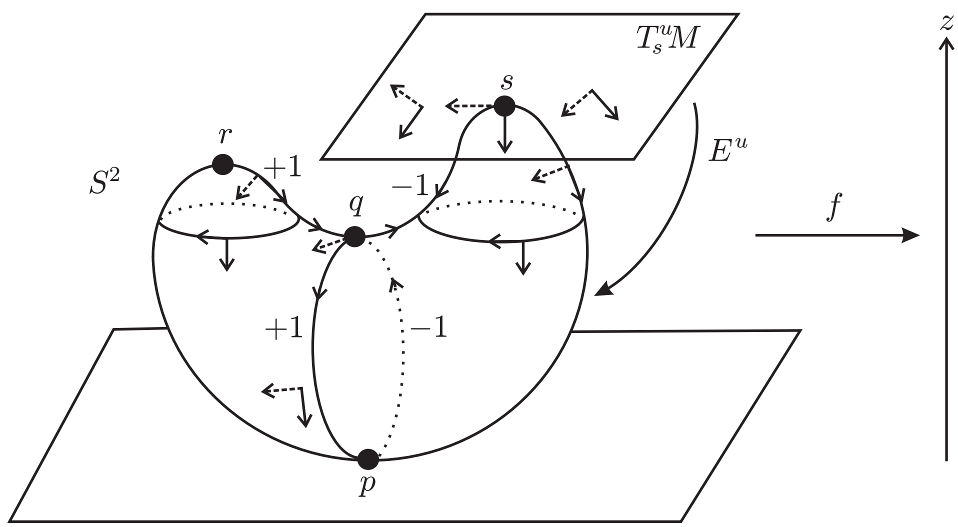

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The project

Construct a (singular) chain complex analogous to the Morse-Smale-Witten chain complex for Morse-Bott functions.

Question: Why would anyone want to do this? After all, we can always perturb a smooth function to get a Morse-Smale function. Also, a Morse-Bott function determines a filtration, and hence, a spectral sequence.

Example

If $\pi : E \to B$ is a smooth fiber bundle with fiber F, and f is a Morse function on B, then $f \circ \pi$ is a Morse-Bott function with critical submanifolds diffeomorphic to F.

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \pi \\ & & B \xrightarrow{f} \mathbb{R} \end{array}$$

In particular, if G is a Lie group acting on M and $\pi : EG \to BG$ is the classifying bundle for G, then

$$\begin{array}{ccc} M & \longrightarrow & EG \times_G M \\ & & \downarrow \pi \\ & & BG \xrightarrow{f} \mathbb{R} \end{array}$$

So, this might be useful for studying equivariant homology: $H_*^G(M) = H_*(EG \times_G M)$.

Other Examples: The square of the moment map, product structures in symplectic Floer homology, quantum cohomology, etc.

Perturbations

1. If $f : M \to \mathbb{R}$ is a Morse-Bott function, study the Morse-Smale-Witten complex as $\varepsilon \to 0$ of

$$h = f + \varepsilon \left(\sum_{j=1}^{l} \rho_j f_j \right).$$

2. If $h : M \to \mathbb{R}$ is a Morse-Smale function, study the Morse-Smale-Witten complex of $\varepsilon h : M \to \mathbb{R}$ as $\varepsilon \to 0$.

Morse-Bott functions

Definition 1 *A smooth function* $f : M \to \mathbb{R}$ *on a smooth manifold* M *is called a Morse-Bott function if and only if* $\mathrm{Cr}(f)$ *is a disjoint union of connected submanifolds, and for each connected submanifold* $B \subseteq \mathrm{Cr}(f)$ *the normal Hessian is non-degenerate for all* $p \in B$.

Lemma 1 (Morse-Bott Lemma) *Let* $f : M \to \mathbb{R}$ *be a Morse-Bott function, and let* B *be a connected component of the critical set* $Cr(f)$*. For any* $p \in B$ *there is a local chart of* M *around* p *and a local splitting of the normal bundle of* B

$$\nu_*(B) = \nu_*^+(B) \oplus \nu_*^-(B)$$

identifying a point $x \in M$ *in its domain to* (u, v, w) *where* $u \in B$, $v \in \nu_*^+(B)$, $w \in \nu_*^-(B)$ *such that within this chart* f *assumes the form*

$$f(x) = f(u, v, w) = f(B) + |v|^2 - |w|^2.$$

Note that if $p \in B$, then this implies that

$$T_pM = T_pB \oplus \nu_p^+(B) \oplus \nu_p^-(B).$$

If we let $\lambda_p = \dim\, \nu_p^-(B)$ be the index of a connected critical submanifold B, $b = \dim\, B$, and $\lambda_p^* = \dim\, \nu_p^+(B)$, then we have the fundamental relation

$$m \;=\; b + \lambda_p^* + \lambda_p$$

where $m = \dim\, M$.

Morse-Bott functions II

For $p \in Cr(f)$ the stable manifold $W^s(p)$ and the unstable manifold $W^u(p)$ are defined the same as they are for a Morse function:

$$W^s(p) = \{x \in M | \lim_{t\to\infty} \varphi_t(x) = p\}$$
$$W^u(p) = \{x \in M | \lim_{t\to-\infty} \varphi_t(x) = p\}.$$

Definition 2 *If $f : M \to \mathbb{R}$ is a Morse-Bott function, then the stable and unstable manifolds of a critical submanifold B are defined to be*

$$W^s(B) = \bigcup_{p\in B} W^s(p)$$
$$W^u(B) = \bigcup_{p\in B} W^u(p).$$

Theorem 1 (Stable/Unstable Manifold Theorem) *The stable and unstable manifolds $W^s(B)$ and $W^u(B)$ are the surjective images of smooth injective immersions $E^+ : \nu_*^+(B) \to M$ and $E^- : \nu_*^-(B) \to M$. There are smooth endpoint maps $\partial_+ : W^s(B) \to B$ and $\partial_- : W^u(B) \to B$ given by $\partial_+(x) = \lim_{t\to\infty} \varphi_t(x)$ and $\partial_-(x) = \lim_{t\to-\infty} \varphi_t(x)$ which when restricted to a neighborhood of B have the structure of locally trivial fiber bundles.*

Morse-Bott-Smale functions

Definition 3 (Morse-Bott-Smale Transversality) *A function $f : M \to \mathbb{R}$ is said to satisfy the* ***Morse-Bott-Smale transversality*** *condition with respect to a given metric on M if and only if f is Morse-Bott and for any two connected critical submanifolds B and B', $W^u(p)$ intersects $W^s(B')$ transversely, i.e. $W^u(p) \pitchfork W^s(B')$, for all $p \in B$.*

Note: For a given Morse-Bott function $f : M \to \mathbb{R}$ it may not be possible to pick a Riemannian metric for which f is Morse-Bott-Smale.

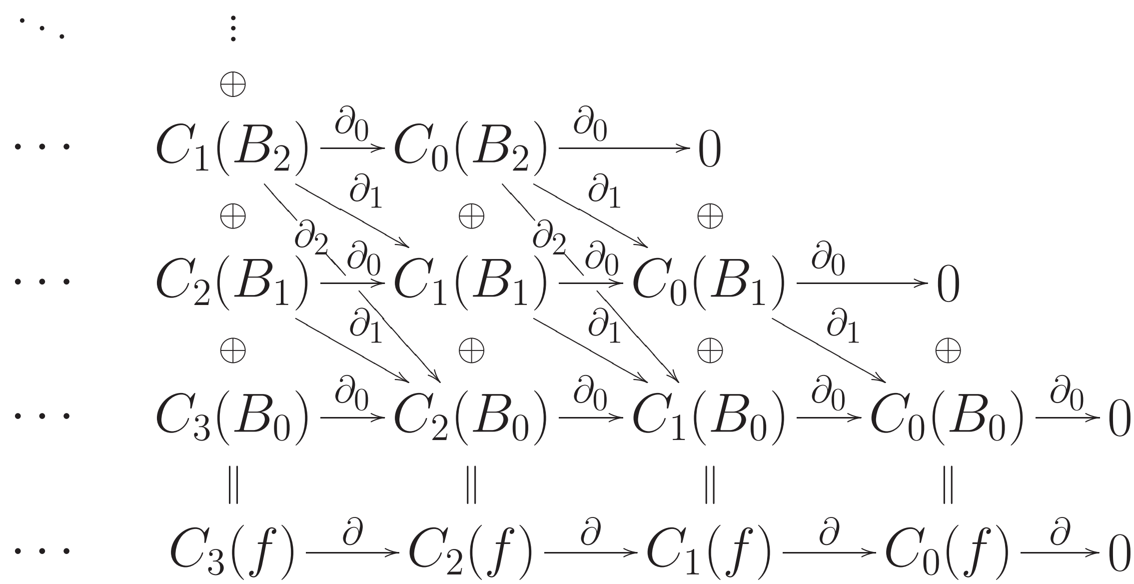
Lemma 2 *Suppose that B is of dimension b and index λ_B and that B' is of dimension b' and index $\lambda_{B'}$. Then we have the following where $m = dim\ M$:*

$$\begin{aligned} dim\, W^u(B) &= b + \lambda_B \\ dim\, W^s(B') &= b' + \lambda^*_{B'} = m - \lambda_{B'} \\ dim\, W(B, B') &= \lambda_B - \lambda_{B'} + b \quad (if\ W(B, B') \neq \emptyset). \end{aligned}$$

Note: The dimension of $W(B, B')$ does not depend on the dimension of the critical submanifold B'. This fact will be used when we define the boundary operator in the Morse-Bott-Smale chain complex.

The general form of a M-B-S complex

Assume that $f : M \to \mathbb{R}$ is a Morse-Bott-Smale function and the manifold M, the critical submanifolds, and their negative normal bundles are all orientable. Let $C_p(B_i)$ be the group of "p-dimensional chains" in the critical submanifolds of index i. A Morse-Bott-Smale chain complex is of the form:



where the boundary operator is defined as a sum of homomorphisms $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ where $\partial_j : C_p(B_i) \to C_{p+j-1}(B_{i-j})$. This type of algebraic structure is known as a multicomplex.

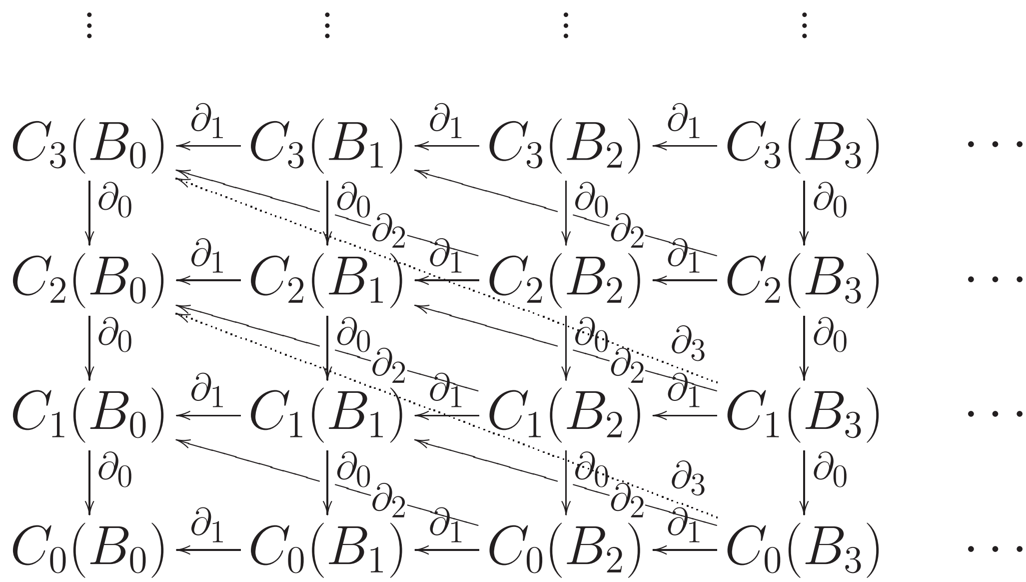
The homomorphism ∂_0: For a deRham-type cohomology theory $\partial_0 = d$. For a singular theory $\partial_0 = (-1)^k \partial$, where ∂ is the "usual" boundary operator from singular homology.

Ways to define $\partial_1, \ldots, \partial_m$:

1. deRham version: integration along the fiber.
2. singular versions: fibered product constructions.

The associated spectral sequence

The Morse-Bott chain multicomplex can be written as follows to resemble a first quadrant spectral sequence.



More precisely, the Morse-Bott chain complex $(C_*(f), \partial)$ is a filtered differential graded $\mathbb{Z}$-module where the (increasing) filtration is determined by the Morse-Bott index. The associated bigraded module $G(C_*(f))$ is given by

$$G(C_*(f))_{s,t} = F_s C_{s+t}(f)/F_{s-1}C_{s+t}(f) \approx C_t(B_s),$$

and the E^1 term of the associated spectral sequence is given by

$$E^1_{s,t} \approx H_{s+t}(F_s C_*(f)/F_{s-1}C_*(f))$$

where the homology is computed with respect to the boundary operator on the chain complex $F_s C_*(f)/F_{s-1}C_*(f)$ induced by $\partial = \partial_0 \oplus \cdots \oplus \partial_m$, i.e. ∂_0.

The associated spectral sequence II

Since $\partial_0 = (-1)^k \partial$, where ∂ is the "usual" boundary operator from singular homology, the E^1 term of the spectral sequence is given by

$$E^1_{s,t} \approx H_{s+t}(F_s C_*(f)/F_{s-1}C_*(f)) \approx H_t(B_s)$$

where $H_t(B_s)$ denotes homology of the chain complex

$$\cdots \xrightarrow{\partial_0} C_3(B_s) \xrightarrow{\partial_0} C_2(B_s) \xrightarrow{\partial_0} C_1(B_s) \xrightarrow{\partial_0} C_0(B_s) \xrightarrow{\partial_0} 0.$$

Hence, the E^1 term of the spectral sequence is

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
H_3(B_0) & \xleftarrow{d_1} & H_3(B_1) & \xleftarrow{d_1} & H_3(B_2) & \xleftarrow{d_1} & H_3(B_3) & & \cdots \\
H_2(B_0) & \xleftarrow{d_1} & H_2(B_1) & \xleftarrow{d_1} & H_2(B_2) & \xleftarrow{d_1} & H_2(B_3) & & \cdots \\
H_1(B_0) & \xleftarrow{d_1} & H_1(B_1) & \xleftarrow{d_1} & H_1(B_2) & \xleftarrow{d_1} & H_1(B_3) & & \cdots \\
H_0(B_0) & \xleftarrow{d_1} & H_0(B_1) & \xleftarrow{d_1} & H_0(B_2) & \xleftarrow{d_1} & H_0(B_3) & & \cdots
\end{array}$$

where d_1 denotes the following connecting homomorphism of the triple $(F_s C_*(f), F_{s-1}C_*(f), F_{s-2}C_*(f)$.

$$H_{s+t}(F_s C_*(f)/F_{s-1}C_*(f)) \xrightarrow{d_1} H_{s+t-1}(F_{s-1}C_*(f)/F_{s-2}C_*(f))$$

The differentials d_0 and d_1 in the spectral sequence are induced from the homomorphisms ∂_0 and ∂_1 in the multicomplex. However, the differential d_r for $r \geq 2$ is, in general, not induced from the corresponding homomorphism ∂_r in the multicomplex [J.M. Boardman, "Conditionally convergent spectral sequences"].

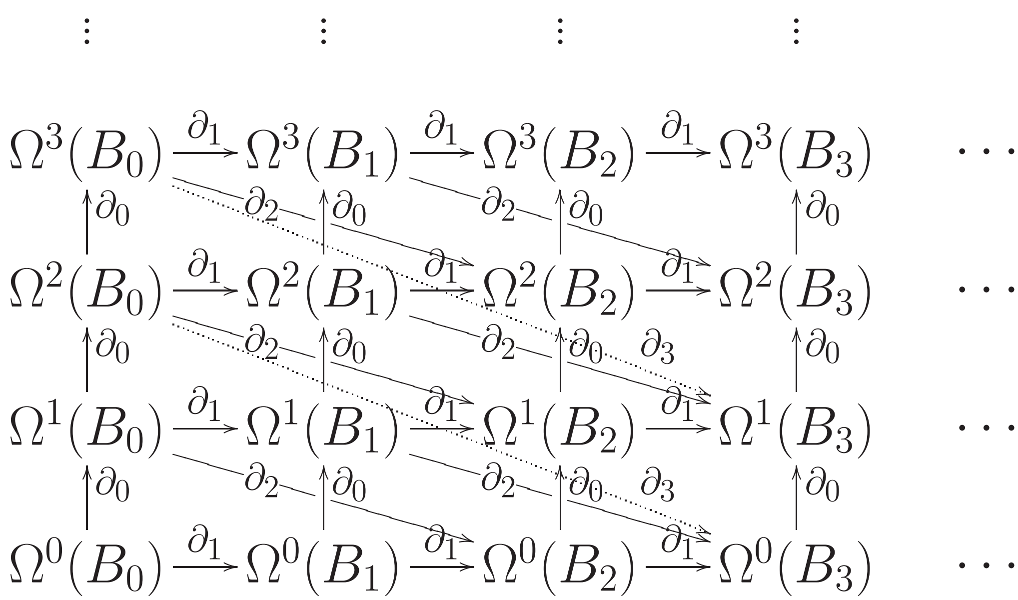
The Austin–Braam approach (∼1995)

(Modeled on deRham cohomology)

Let B_i be the set of critical points of index i and $C^{i,j} = \Omega^j(B_i)$ the set of j-forms on B_i. Austin and Braam define maps

$$\partial_r : C^{i,j} \to C^{i+r,j-r+1}$$

for $r = 0, 1, 2, \ldots, m$ which raise the "total degree" $i + j$ by one.

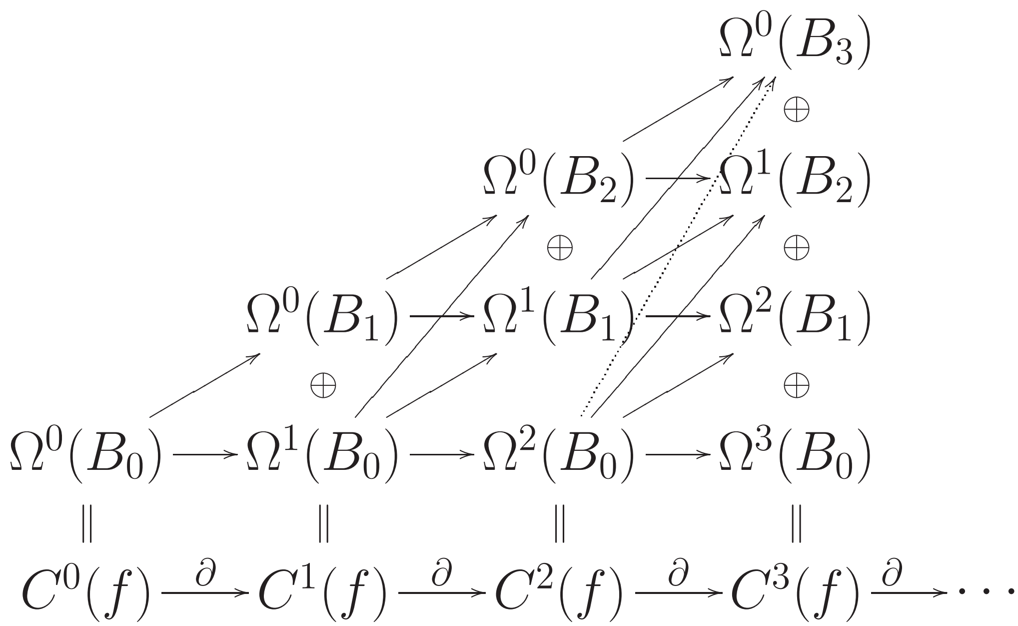


Note: Note that the above diagram is not a double complex because $\partial_1^2 \neq 0$. However, it does determine a multicomplex [J.-P. Meyer, "Acyclic models for multicomplexes", Duke Math. J., **45** (1978), no. 1, p. 67–85; MR 0486489 (80b:55012)].)

The Austin–Braam cochain complex

The maps $\partial_r : \Omega^j(B_i) \longrightarrow \Omega^{j-r+1}(B_{i+r})$ fit together to form a cochain complex where $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ and

$$C^k(f) = \bigoplus_{i=0}^{k} \Omega^{k-i}(B_i).$$



Theorem 2 (Austin-Braam) *For any* $j = 0, \ldots, m$

$$\sum_{l=0}^{j} \partial_{j-l}\partial_j = 0.$$

Hence, $\partial^2 = 0$.

Note: $\partial_2\partial_0 + \partial_1\partial_1 + \partial_0\partial_2 = 0$. So, $\partial_1^2 \neq 0$ in general.

Theorem 3 (Austin-Braam)

$$H(C^*(f), \partial) \approx H^*(M; \mathbb{R})$$

Compactified moduli spaces

For any two critical submanifolds B and B' the flow φ_t induces an $\mathbb{R}$-action on $W^u(B) \cap W^s(B')$. Let

$$\mathcal{M}(B, B') = (W^u(B) \cap W^s(B'))/\mathbb{R}$$

be the quotient space of gradient flow lines from B to B'.

Theorem 4 (Gluing) *Suppose that B, B', and B'' are critical submanifolds such that $W^u(B) \pitchfork W^s(B')$ and $W^u(B') \pitchfork W^s(B'')$. In addition, assume that $W^u(x) \pitchfork W^s(B'')$ for all $x \in B'$. Then for some $\epsilon > 0$, there is an injective local diffeomorphism*

$$G : \mathcal{M}(B, B') \times_{B'} \mathcal{M}(B', B'') \times (0, \epsilon) \to \mathcal{M}(B, B'')$$

onto an end of $\mathcal{M}(B, B'')$.

Theorem 5 (Compactification) *Assume that $f : M \to \mathbb{R}$ satisfies the Morse-Bott-Smale transversality condition. For any two distinct critical submanifolds B and B' the moduli space $\mathcal{M}(B, B')$ has a compactification $\overline{\mathcal{M}}(B, B')$, consisting of all the piecewise gradient flow lines from B to B', which is a compact smooth manifold with corners of dimension $\lambda_B - \lambda_{B'} + b - 1$. Moreover, the beginning and endpoint maps extend to smooth maps*

$$\partial_- : \overline{\mathcal{M}}(B, B') \to B$$
$$\partial_+ : \overline{\mathcal{M}}(B, B') \to B',$$

where ∂_- has the structure of a locally trivial fiber bundle.

Integration along the fiber

Let $\pi : E \to B$ be a fiber bundle where B is a closed manifold, a typical fiber F is a compact oriented d-dimensional manifold with corners, and $\pi_\partial : \partial E \to B$ is also a fiber bundle with fiber ∂F. A differential form on E may be written locally as

$$\pi^*(\phi) f(x,t) dt_{i_1} \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_r}$$

where ϕ is a form on B, x are coordinates on B, and the t_j are coordinates on F.

Definition 4 *Integration along the fiber*

$$\pi_* : \Omega^j(E) \to \Omega^{j-d}(B)$$

is defined by

$$\begin{aligned} \pi_*(\pi^*(\phi) f(x,t) dt_1 \wedge dt_2 \wedge \cdots \wedge dt_d) &= \phi \int_F f(x,t) dt_1 \wedge \cdots \wedge dt_d \\ \pi_*(\pi^*(\phi) f(x,t) dt_{i_1} \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_r}) &= 0 \quad \text{if } r < d. \end{aligned}$$

The beginning point map

$$\partial_- : \overline{M}(B_{i+r}, B_i) \to B_{i+r}$$

is such a fiber bundle and we can pullback along the endpoint map

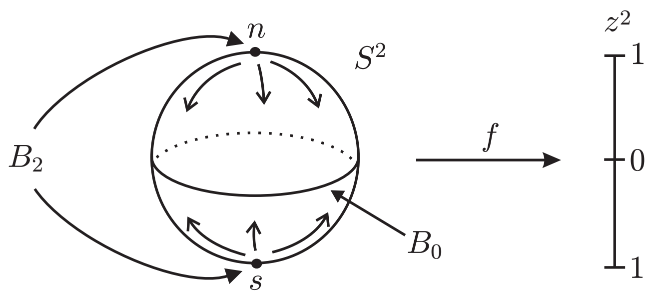
$$\partial_+ : \overline{M}(B_{i+r}, B_i) \to B_i.$$

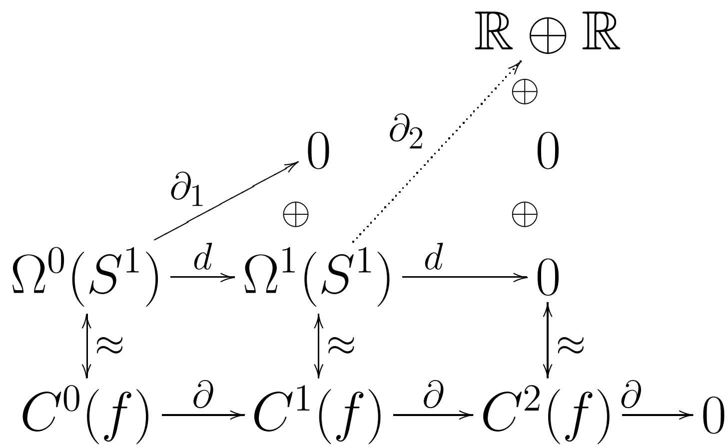
Definition 5 *Define* $\partial_r : \Omega^j(B_i) \to \Omega^{j-r+1}(B_{i+r})$ *by*

$$\partial_r(\omega) = \begin{cases} d\omega & r = 0 \\ (-1)^j (\partial_-)_* (\partial_+^* \omega) & r \neq 0. \end{cases}$$

An example of Morse-Bott cohomology

Consider $S^2 = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}$, and let $f(x, y, z) = z^2$. Then $B_0 = E \approx S^1$, $B_1 = \emptyset$, and $B_2 = \{n, s\}$.





$$\ker d : \Omega^0(S^1) \to \Omega^1(S^1) \approx \text{constant functions on } S^1$$
$$\approx H^0(S^2; \mathbb{R})$$

The map $\partial_2 : \Omega^1(S^1) \to \mathbb{R} \oplus \mathbb{R}$ integrates a 1-form ω over the components of $\overline{\mathcal{M}}(B_2, B_0) \approx S^1 \amalg S^1$, which have opposite orientations. So,

$$\partial_2(\omega) = (-1)(\partial_-)_*(\partial_+^*\omega) = (c, -c)$$

for some $c \in \mathbb{R}$, and $H^2(C^*(f), \partial) \approx \mathbb{R}^2/\mathbb{R} \approx \mathbb{R}$. If $c = 0$, then ω is in the image of $d : \Omega^0(S^1) \to \Omega^1(S^1)$, and hence $H^1(C^*(f), \partial) \approx 0$.

The Banyaga–Hurtubise approach (∼2007)

Modeled on cubical singular homology. Based on ideas from Austin and Braam (∼1995), Barraud and Cornea (∼2004), Fukaya (∼1995), Weber (∼2006) etc.

Step 1: Generalize the notion of singular p-simplexes to allow maps from spaces other than the standard p-simplex $\triangle^p \subset \mathbb{R}^{p+1}$ or the unit p-cube $I^p \subset \mathbb{R}^p$. These generalizations of $\triangle^p$ (or I^p) are called abstract topological chains, and the corresponding singular chains are called singular topological chains.

Step 2: Show that the compactified moduli spaces of gradient flow lines are abstract topological chains, i.e. show that ∂_0 is defined. Show that ∂_0 extends to fibered products.

Step 3: Define the set of allowed domains C_p in the Morse-Bott-Smale chain complex as a collection of fibered products (with ∂_0 defined) and show that the allowed domains are all compact oriented smooth manifolds with corners.

Step 4: Define $\partial_1, \ldots, \partial_m$ using fibered products of compactified moduli spaces of gradient flow lines and the beginning and endpoint maps. Define $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ and show that $\partial \circ \partial = 0$.

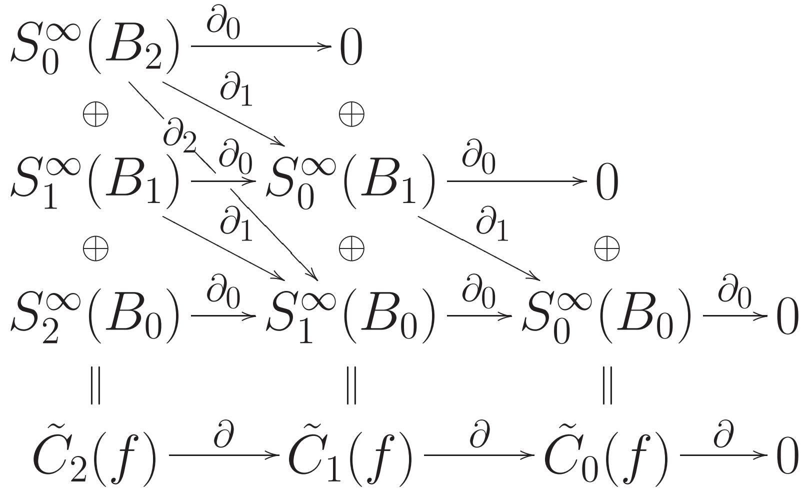
Step 5: Define orientation conventions on the elements of C_p and corresponding degeneracy relations to identify singular topological chains that are "essentially" the same. Show that $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ is compatible with the degeneracy relations.

Step 6: Show that the homology of the Morse-Bott-Smale chain complex $(C_*(f), \partial_*)$ is independent of $f : M \to \mathbb{R}$.

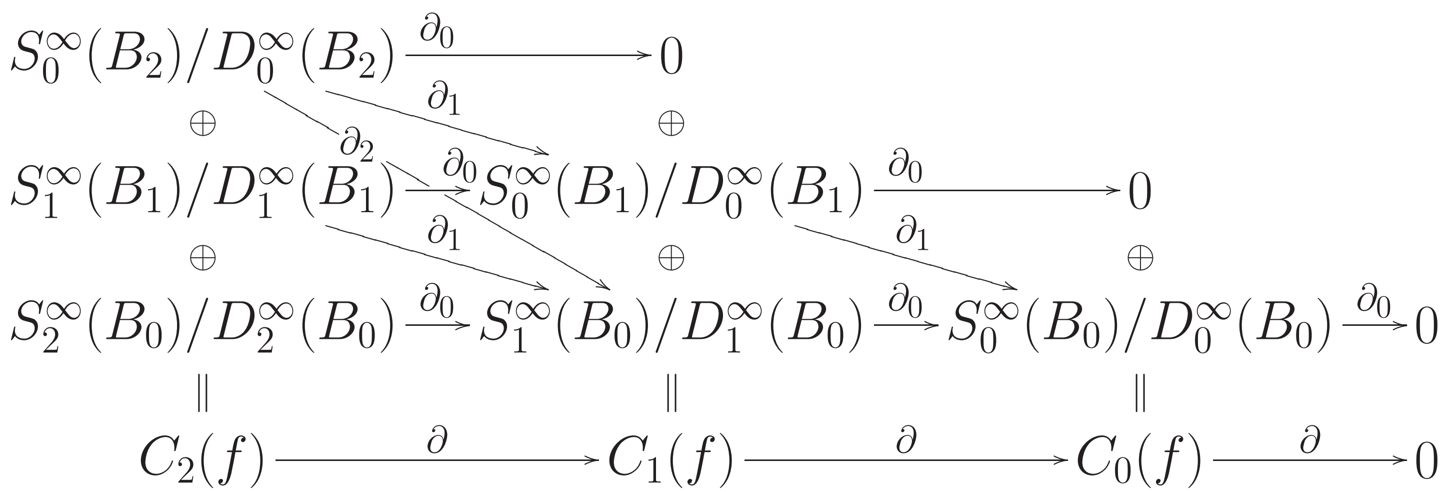
The singular M-B-S chain complex

Let $S_p^\infty(B_i)$ be the set of smooth singular C_p-chains in B_i (with respect to the endpoint maps on moduli spaces), and let $D_p^\infty(B_i) \subseteq S_p^\infty(B_i)$ be the subgroup of degenerate singular topological chains.

The chain complex $(\tilde{C}_*(f), \partial)$:



The Morse-Bott-Smale chain complex $(C_*(f), \partial)$:



Step 1: Generalize the notion of singular p-simplexes to allow maps from spaces other than the standard p-simplex $\triangle^p \subset \mathbb{R}^{p+1}$ or the unit p-cube $I^p \subset \mathbb{R}^p$.

For each integer $p \geq 0$ fix a set C_p of topological spaces, and let S_p be the free abelian group generated by the elements of C_p, i.e. $S_p = \mathbb{Z}[C_p]$. Set $S_p = \{0\}$ if $p < 0$ or $C_p = \emptyset$.

Definition 6 *A boundary operator on the collection S_* of groups $\{S_p\}$ is a homomorphism $\partial_p : S_p \to S_{p-1}$ such that*

1. *For $p \geq 1$ and $P \in C_p \subseteq S_p$, $\partial_p(P) = \sum_k n_k P_k$ where $n_k = \pm 1$ and $P_k \in C_{p-1}$ is a subspace of P for all k.*
2. *$\partial_{p-1} \circ \partial_p : S_p \to S_{p-2}$ is zero.*

We call (S_*, ∂_*) a chain complex of abstract topological chains. Elements of S_p are called abstract topological chains of degree p.

Definition 7 *Let B be a topological space and $p \in \mathbb{Z}_+$. A singular C_p-space in B is a continuous map $\sigma : P \to B$ where $P \in C_p$, and the singular C_p-chain group $S_p(B)$ is the free abelian group generated by the singular C_p-spaces. Define $S_p(B) = \{0\}$ if $S_p = \{0\}$ or $B = \emptyset$. Elements of $S_p(B)$ are called singular topological chains of degree p.*

Note: These definitions are quite general. To construct the M-B-S chain complex we really only need C_p to include the p-dimensional faces of an N-cube, the compactified moduli spaces of gradient flow lines of dimension p, and the components of their fibered products of dimension p.

For $p \geq 1$ there is a boundary operator $\partial_p : S_p(B) \longrightarrow S_{p-1}(B)$ induced from the boundary operator $\partial_p : S_p \longrightarrow S_{p-1}$. If $\sigma : P \longrightarrow B$ is a singular C_p-space in B, then $\partial_p(\sigma)$ is given by the formula

$$\partial_p(\sigma) = \sum_k n_k \sigma|_{P_k}$$

where

$$\partial_p(P) = \sum_k n_k P_k.$$

The pair $(S_*(B), \partial_*)$ is called a chain complex of singular topological chains in B.

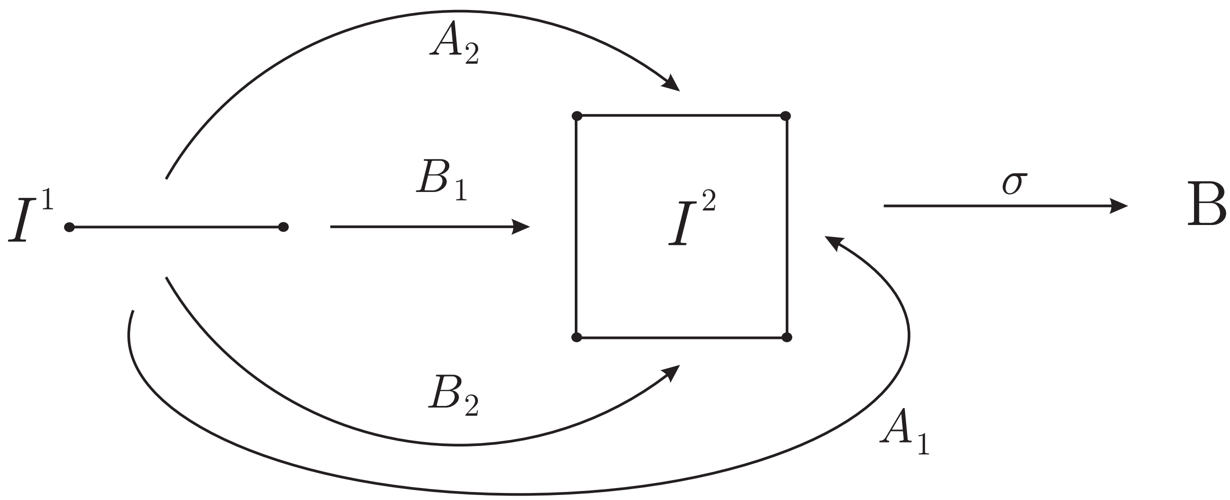
Singular N-cube chains

Pick some large positive integer N and let $I^N = \{(x_1, \ldots, x_N) \in \mathbb{R}^N |\ 0 \leq x_j \leq 1,\ j = 1, \ldots, N\}$ denote the unit N-cube. For every $0 \leq p \leq N$ let C_p be the set consisting of the faces of I^N of dimension p, i.e. subsets of I^N where p of the coordinates are free and the rest of the coordinates are fixed to be either 0 or 1. For every $0 \leq p \leq N$ let S_p be the free abelian group generated by the elements of C_p. For $P \in C_p$ we define

$$\partial_p(P) = \sum_{j=1}^{p} (-1)^j \left[P|_{x_j=1} - P|_{x_j=0} \right] \in S_{p-1}$$

where x_j denotes the j^{th} free coordinate of P.

Singular cubical boundary operator (Massey)

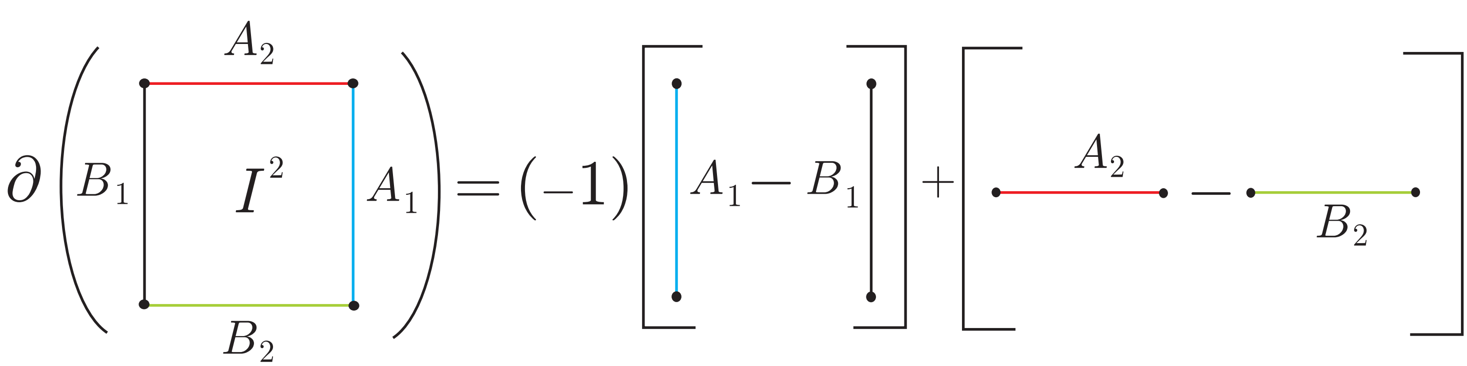


The chain $\sigma : I^2 \to B$ has boundary

$$\partial_2(\sigma) = (-1)[\sigma \circ A_1 - \sigma \circ B_1] + [\sigma \circ A_2 - \sigma \circ B_2]$$

where the terms in the sum are all maps with domain $I^1 = [0, 1]$.

Topological cubical boundary operator (B–H)



The chain $\sigma : I^2 \to B$ has boundary

$$\partial_2(\sigma) = (-1)[\sigma|_{A_1} - \sigma|_{B_1}] + [\sigma|_{A_2} - \sigma|_{B_2}]$$

and the degeneracy relations identify terms that are "essentially" the same.

Recovering singular homology (degeneracy relations)

A continuous map $\sigma_P : P \to B$ from a p-face P of I^N into a topological space B is a singular C_p-space in B. The boundary operator applied to σ_P is

$$\partial_p(\sigma_P) = \sum_{j=1}^{p} (-1)^j \left[\sigma_P|_{x_j=1} - \sigma_P|_{x_j=0}\right] \in S_{p-1}(B)$$

where $\sigma_P|_{x_j=0}$ denotes the restriction $\sigma_P : P|_{x_j=0} \to B$ and $\sigma_P|_{x_j=1}$ denotes the restriction $\sigma_P : P|_{x_j=1} \to B$.

Definition 8 *Let σ_P and σ_Q be singular C_p-spaces in B and let $\partial_p(Q) = \sum_j n_j Q_j \in S_{p-1}$. For any map $\alpha : P \to Q$, let $\partial_p(\sigma_Q) \circ \alpha$ denote the formal sum $\sum_j n_j(\sigma_Q \circ \alpha)|_{\alpha^{-1}(Q_j)}$. Define the subgroup $D_p(B) \subseteq S_p(B)$ of degenerate singular N-cube chains to be the subgroup generated by the following elements.*

1. *If α is an orientation preserving homeomorphism such that $\sigma_Q \circ \alpha = \sigma_P$ and $\partial_p(\sigma_Q) \circ \alpha = \partial_p(\sigma_P)$, then $\sigma_P - \sigma_Q \in D_p(B)$.*
2. *If σ_P does not depend on some free coordinate of P, then $\sigma_P \in D_p(B)$.*

Theorem 6 *The boundary operator for singular N-cube chains $\partial_p : S_p(B) \to S_{p-1}(B)$ descends to a homomorphism*

$$\partial_p : S_p(B)/D_p(B) \to S_{p-1}(B)/D_{p-1}(B),$$

and

$$H_p(S_*(B)/D_*(B), \partial_*) \approx H_p(B; \mathbb{Z})$$

for all $p < N$.

Step 2: Show that the compactified moduli spaces of gradient flow lines are abstract topological chains, i.e. show that ∂_0 is defined. Show that ∂_0 extends to fibered products.

Fibered products

Suppose that $\sigma_1 : P_1 \to B$ is a singular S_{p_1}-space and $\sigma_2 : P_2 \to B$ is a singular S_{p_2}-space where (S_*, ∂_*) is a chain complex of abstract topological chains. The fibered product of σ_1 and σ_2 is

$$P_1 \times_B P_2 = \{(x_1, x_2) \in P_1 \times P_2 |\ \sigma_1(x_1) = \sigma_2(x_2)\}.$$

This construction extends linearly to singular topological chains.

Definition 9 *The degree of the fibered product $P_1 \times_B P_2$ is defined to be $p_1 + p_2 - b$. The boundary operator applied to the fibered product is defined to be*

$$\partial(P_1 \times_B P_2) = \partial P_1 \times_B P_2 + (-1)^{p_1+b} P_1 \times_B \partial P_2$$

where ∂P_1 and ∂P_2 denote the boundary operator applied to the abstract topological chains P_1 and P_2. If $\sigma_1 = 0$, then we define $0 \times_B P_2 = 0$. Similarly, if $\sigma_2 = 0$, then $P_1 \times_B 0 = 0$.

Lemma 3 *The fibered product of two singular topological chains is an abstract topological chain, i.e. the boundary operator on fibered products is of degree -1 and satisfies $\partial \circ \partial = 0$. Moreover, the boundary operator on fibered products is associative, i.e.*

$$\partial((P_1 \times_{B_1} P_2) \times_{B_2} P_3) = \partial(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)).$$

Proof that $P_1 \times_B P_2$ is an abstract topological chain

The degree of $P_1 \times_B P_2$ is $p_1 + p_2 - b$.

Since ∂ is a boundary operator on P_1 and P_2, the degree of ∂P_1 is $p_1 - 1$ and the degree of ∂P_2 is $p_2 - 1$. Hence both $\partial P_1 \times_B P_2$ and $P_1 \times_B \partial P_2$ have degree $p_1 + p_2 - b - 1$.

To see that $\partial^2(P_1 \times_B P_2) = 0$ we compute as follows.

$$\begin{aligned}
\partial(\partial(P_1 \times_B P_2)) &= \partial(\partial P_1 \times_B P_2 + (-1)^{p_1+b} P_1 \times_B \partial P_2) \\
&= \partial^2 P_1 \times_B P_2 + (-1)^{p_1-1+b} \partial P_1 \times_B \partial P_2 + \\
&\qquad (-1)^{p_1+b}(\partial P_1 \times_B \partial P_2 + (-1)^{p_1+b} P_1 \times_B \partial^2 P_2) \\
&= 0.
\end{aligned}$$

Associativity

Given the data of a triple

$$P_1 \xrightarrow{\sigma_{11}} B_1 \xleftarrow{\sigma_{12}} P_2 \xrightarrow{\sigma_{22}} B_2 \xleftarrow{\sigma_{23}} P_3$$

we can form the iterated fibered product $(P_1 \times_{B_1} P_2) \times_{B_2} P_3$ using σ_{23} and the map $\sigma_{22} \circ \pi_2 : P_1 \times_{B_1} P_2 \to B_2$, where $\pi_2 : P_1 \times_{B_1} P_2 \to P_2$ denotes projection to the second component. That is, we have the following diagram.

$$\begin{array}{ccccc}
(P_1 \times_{B_1} P_2) \times_{B_2} P_3 & \overset{\pi_3}{\dashrightarrow} & & & P_3 \\
\downarrow \pi_1 & & & & \downarrow \sigma_{23} \\
P_1 \times_{B_1} P_2 & \overset{\pi_2}{\dashrightarrow} & P_2 & \xrightarrow{\sigma_{22}} & B_2 \\
\downarrow \pi_1 & & \downarrow \sigma_{12} & & \\
P_1 & \xrightarrow{\sigma_{11}} & B_1 & &
\end{array}$$

Compactified moduli spaces and ∂_0

Definition 10 *Let B_i be the set of critical points of index i. For any $j = 1, \ldots, i$ we define the degree of $\overline{\mathcal{M}}(B_i, B_{i-j})$ to be $j + b_i - 1$ and the boundary operator to be*

$$\partial\overline{\mathcal{M}}(B_i, B_{i-j}) = (-1)^{i+b_i} \sum_{i-j<n<i} \overline{\mathcal{M}}(B_i, B_n) \times_{B_n} \overline{\mathcal{M}}(B_n, B_{i-j})$$

where $b_i = \dim B_i$ and the fibered product is taken over the beginning and endpoint maps ∂_- and ∂_+. If $B_n = \emptyset$, then $\overline{\mathcal{M}}(B_i, B_n) = \overline{\mathcal{M}}(B_n, B_{i-j}) = 0$.

Lemma 4 *The degree and boundary operator for $\overline{\mathcal{M}}(B_i, B_{i-j})$ satisfy the axioms for abstract topological chains, i.e. the boundary operator on the compactified moduli spaces is of degree -1 and $\partial \circ \partial = 0$.*

Proof: Let $d = \deg \overline{\mathcal{M}}(B_i, B_n) = i - n + b_i - 1$. Then $\partial(\overline{\mathcal{M}}(B_i, B_n) \times_{B_n} \overline{\mathcal{M}}(B_n, B_{i-j}))$

$$\begin{aligned} &= \partial\overline{\mathcal{M}}(B_i, B_n) \times_{B_n} \overline{\mathcal{M}}(B_n, B_{i-j}) + (-1)^{d+b_n}\overline{\mathcal{M}}(B_i, B_n) \times_{B_n} \partial\overline{\mathcal{M}}(B_n, B_{i-j}) \\ &= (-1)^{i+b_i} \sum_{n<s<i} \overline{\mathcal{M}}(B_i, B_s, B_n, B_{i-j}) + (-1)^{i+b_i-1} \sum_{i-j<t<n} \overline{\mathcal{M}}(B_i, B_n, B_t, B_{i-j}) \end{aligned}$$

Therefore,

$$\begin{aligned} \partial^2\overline{\mathcal{M}}(B_i, B_{i-j}) &= (-1)^{i+b_i} \left[\sum_{i-j<n<i} \left((-1)^{i+b_i} \sum_{n<s<i} \overline{\mathcal{M}}(B_i, B_s, B_n, B_{i-j}) + \right.\right. \\ &\qquad \left.\left. (-1)^{i+b_i-1} \sum_{i-j<t<n} \overline{\mathcal{M}}(B_i, B_n, B_t, B_{i-j}) \right) \right] \\ &= (-1)^{i+b_i} \left[(-1)^{i+b_i} \sum_{i-j<n<s<i} \overline{\mathcal{M}}(B_i, B_s, B_n, B_{i-j}) + \right. \\ &\qquad \left. (-1)^{i+b_i-1} \sum_{i-j<t<n<i} \overline{\mathcal{M}}(B_i, B_n, B_t, B_{i-j}) \right] \\ &= 0 \end{aligned}$$

□

Step 3: Define the set of allowed domains C_p in the Morse-Bott-Smale chain complex as a collection of fibered products (with ∂_0 defined) and show that the allowed domains are all compact oriented smooth manifolds with corners.

For any $p \geq 0$ let C_p be the set consisting of the faces of I^N of dimension p and the connected components of degree p of fibered products of the form

$$Q \times_{B_{i_1}} \overline{\mathcal{M}}(B_{i_1}, B_{i_2}) \times_{B_{i_2}} \overline{\mathcal{M}}(B_{i_2}, B_{i_3}) \times_{B_{i_3}} \cdots \times_{B_{i_{n-1}}} \overline{\mathcal{M}}(B_{i_{n-1}}, B_{i_n})$$

where $m \geq i_1 > i_2 > \cdots > i_n \geq 0$, Q is a face of I^N of dimension $q \leq p$, $\sigma : Q \to B_{i_1}$ is smooth, and the fibered products are taken with respect to σ and the beginning and endpoint maps.

Theorem 7 *The elements of C_p are compact oriented smooth manifolds with corners, and there is a boundary operator*

$$\partial : S_p \to S_{p-1}$$

where S_p is the free abelian group generated by the elements of C_p.

Let $S_p^\infty(B_i)$ denote the subgroup of the singular C_p-chain group $S_p(B_i)$ generated by those maps $\sigma : P \to B_i$ that satisfy the following two conditions:

1. The map σ is smooth.
2. If $P \in C_p$ is a connected component of a fibered product, then $\sigma = \partial_+ \circ \pi$, where π denotes projection onto the last component of the fibered product.

Define $\partial_0 : S_p^\infty(B_i) \to S_{p-1}^\infty(B_i)$ by $\partial_0 = (-1)^{p+i}\partial$.

Step 4: Define $\partial_1, \ldots, \partial_m$ using fibered products of compactified moduli spaces of gradient flow lines and the beginning and endpoint maps. Define $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ and show that $\partial \circ \partial = 0$.

If $\sigma : P \to B_i$ is a singular C_p-space in $S_p^\infty(B_i)$, then for any $j = 1, \ldots, i$ composing the projection map π_2 onto the second component of $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ with the endpoint map $\partial_+ : \overline{\mathcal{M}}(B_i, B_{i-j}) \to B_{i-j}$ gives a map

$$P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\pi_2} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\partial_+} B_{i-j}.$$

The next lemma shows that restricting this map to the connected components of the fibered product $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ and adding these restrictions (with the sign determined by the orientation when the dimension of a component is zero) defines an element $\partial_j(\sigma) \in S_{p+j-1}^\infty(B_{i-j})$.

Lemma 5 *If $\sigma : P \to B_i$ is a singular C_p-space in $S_p^\infty(B_i)$, then for any $j = 1, \ldots, i$ adding the components of $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ (with sign when the dimension of a component is zero) yields an abstract topological chain of degree $p + j - 1$. That is, we can identify*

$$P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \in S_{p+j-1}.$$

Thus, for all $j = 1, \ldots, i$ there is an induced homomorphism

$$\partial_j : S_p^\infty(B_i) \to S_{p+j-1}^\infty(B_{i-j})$$

which decreases the Morse-Bott degree $p + i$ by 1.

Proposition 1 *For every* $j = 0, \ldots, m$

$$\sum_{q=0}^{j} \partial_q \partial_{j-q} = 0.$$

Proof: When $q = 0$ we compute as follows.
$\partial_0(\partial_j(P)) = \partial_0 \left(P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \right)$

$$\begin{aligned} &= (-1)^{p+i-1} \left(\partial P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) + (-1)^{p+b_i} P \times_{B_i} \partial \overline{\mathcal{M}}(B_i, B_{i-j}) \right) \\ &= (-1)^{p+i-1} \partial P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \; + \\ &\qquad (-1)^{2p+2b_i+2i-1} \sum_{i-j<n<i} P \times_{B_i} \overline{\mathcal{M}}(B_i, B_n) \times_{B_n} \overline{\mathcal{M}}(B_n, B_{i-j}) \end{aligned}$$

If $1 \leq q \leq j-1$, then

$$\partial_q(\partial_{j-q}(P)) = P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j+q}) \times_{B_{i-j+q}} \overline{\mathcal{M}}(B_{i-j+q}, B_{i-j})$$

and if $q = j$, then

$$\partial_j(\partial_0(P)) = (-1)^{p+i} \partial P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}).$$

Summing these expressions gives the desired result.

$\square$

Corollary 1 *The pair* $(\tilde{C}_*(f), \partial)$ *is a chain complex, i.e.* $\partial \circ \partial = 0$.

Step 5: Define orientation conventions on the elements of C_p and corresponding degeneracy relations to identify singular topological chains that are "essentially" the same. Show that $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ is compatible with the degeneracy relations.

Definition 11 (Degeneracy Relations for the Morse-Bott-Smale Chain Complex) *Let $\sigma_P, \sigma_Q \in S_p^\infty(B_i)$ be singular C_p-spaces in B_i and let $\partial Q = \sum_j n_j Q_j \in S_{p-1}$. For any map $\alpha : P \to Q$, let $\partial_0 \sigma_Q \circ \alpha$ denote the formal sum $(-1)^{p+i} \sum_j n_j (\sigma_Q \circ \alpha)|_{\alpha^{-1}(Q_j)}$. Define the subgroup $D_p^\infty(B_i) \subseteq S_p^\infty(B_i)$ of **degenerate singular topological chains** to be the subgroup generated by the following elements.*

1. *If α is an orientation preserving homeomorphism such that $\sigma_Q \circ \alpha = \sigma_P$ and $\partial_0 \sigma_Q \circ \alpha = \partial_0 \sigma_P$, then $\sigma_P - \sigma_Q \in D_p^\infty(B_i)$.*
2. *If P is a face of I^N and σ_P does not depend on some free coordinate of P, then $\sigma_P \in D_p^\infty(B_i)$ and $\partial_j(\sigma_P) \in D_{p+j-1}^\infty(B_{i-j})$ for all $j = 1, \ldots, m$.*
3. *If P and Q are connected components of some fibered products and α is an orientation reversing map such that $\sigma_Q \circ \alpha = \sigma_P$ and $\partial_0 \sigma_Q \circ \alpha = \partial_0 \sigma_P$, then $\sigma_P + \sigma_Q \in D_p^\infty(B_i)$.*
4. *If Q is a face of I^N and R is a connected component of a fibered product*
$$Q \times_{B_{i_1}} \overline{\mathcal{M}}(B_{i_1}, B_{i_2}) \times_{B_{i_2}} \overline{\mathcal{M}}(B_{i_2}, B_{i_3}) \times_{B_{i_3}} \cdots \times_{B_{i_{n-1}}} \overline{\mathcal{M}}(B_{i_{n-1}}, B_{i_n})$$
such that deg $R >$ dim B_{i_n}, then $\sigma_R \in D_r^\infty(B_{i_n})$ and $\partial_j(\sigma_R) \in D_{r+j-1}^\infty(B_{i_n - j})$ for all $j = 0, \ldots, m$.
5. *If $\sum_\alpha n_\alpha \sigma_\alpha \in S_*(R)$ is a smooth singular chain in a connected component R of a fibered product (as in (4)) that represents the fundamental class of R and*
$$(-1)^{r+i_n} \partial_0 \sigma_R - \sum_\alpha n_\alpha \partial(\sigma_R \circ \sigma_\alpha)$$
is in the group generated by the elements satisfying one of the above conditions, then
$$\sigma_R - \sum_\alpha n_\alpha (\sigma_R \circ \sigma_\alpha) \in D_r^\infty(B_{i_n})$$
and
$$\partial_j \left(\sigma_R - \sum_\alpha n_\alpha (\sigma_R \circ \sigma_\alpha) \right) \in D_{r+j-1}^\infty(B_{i_n - j})$$
for all $j = 1, \ldots, m$.

Step 6: Show that the homology of the Morse-Bott-Smale chain complex $(C_*(f), \partial_*)$ is independent of $f : M \longrightarrow \mathbb{R}$.

Given two Morse-Bott-Smale functions $f_1, f_2 : M \longrightarrow \mathbb{R}$ we pick a smooth function $F_{21} : M \times \mathbb{R} \longrightarrow \mathbb{R}$ meeting certain transversality requirements such that

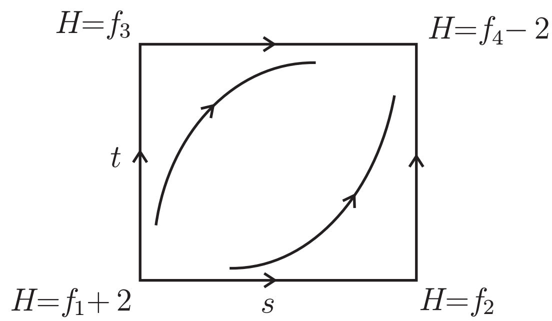
$$\begin{aligned} \lim_{t \to -\infty} F_{21}(x,t) &= f_1(x) + 1 \\ \lim_{t \to +\infty} F_{21}(x,t) &= f_2(x) - 1 \end{aligned}$$

for all $x \in M$. The compactified moduli spaces of gradient flow lines of F_{21} (the *time dependent* gradient flow lines) are used to define a chain map $(F_{21})_\square : C_*(f_1) \longrightarrow C_*(f_2)$, where $(C_*(f_k), \partial)$ is the Morse-Bott chain complex of f_k for $k = 1, 2$.

Next we consider the case where we have four Morse-Bott-Smale functions $f_k : M \longrightarrow \mathbb{R}$ where $k = 1, 2, 3, 4$, and we pick a smooth function $H : M \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ meeting certain transversality requirements such that

$$\begin{aligned} \lim_{s \to -\infty} \lim_{t \to -\infty} H(x,s,t) &= f_1(x) + 2 \\ \lim_{s \to +\infty} \lim_{t \to -\infty} H(x,s,t) &= f_2(x) \\ \lim_{s \to -\infty} \lim_{t \to +\infty} H(x,s,t) &= f_3(x) \\ \lim_{s \to +\infty} \lim_{t \to +\infty} H(x,s,t) &= f_4(x) - 2 \end{aligned}$$

for all $x \in M$.



The compactified moduli spaces of gradient flow lines of H are used to define a chain homotopy between $(F_{43})_\Box \circ (F_{31})_\Box$ and $(F_{42})_\Box \circ (F_{21})_\Box$ where $(F_{lk})_\Box : C_*(f_k) \to C_*(f_l)$ is the map defined above for $k, l = 1, 2, 3, 4$. In homology the map $(F_{kk})_* : H_*(C_*(f_k), \partial) \to H_*(C_*(f_k), \partial)$ is the identity for all k, and hence

$$\begin{aligned}(F_{12})_* \circ (F_{21})_* &= (F_{11})_* \circ (F_{11})_* = id \\ (F_{21})_* \circ (F_{12})_* &= (F_{22})_* \circ (F_{22})_* = id.\end{aligned}$$

Therefore,

$$(F_{21})_* : H_*(C_*(f_1), \partial) \to H_*(C_*(f_2), \partial)$$

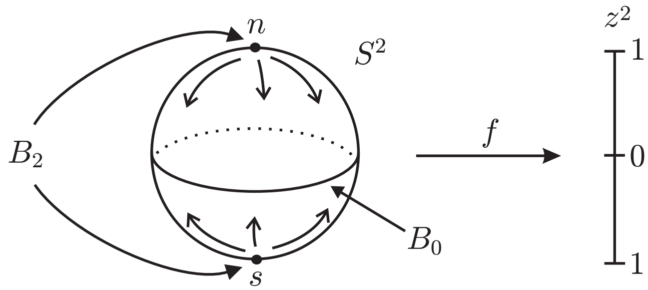
is an isomorphism.

Theorem 8 (Morse-Bott Homology Theorem) *The homology of the Morse-Bott chain complex $(C_*(f), \partial)$ is independent of the Morse-Bott-Smale function $f : M \to \mathbb{R}$. Therefore,*

$$H_*(C_*(f), \partial) \approx H_*(M; \mathbb{Z}).$$

An example of Morse-Bott homology

Consider $M = S^2 = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}$, and let $f(x, y, z) = z^2$. Then $B_0 \approx S^1$, $B_1 = \emptyset$, and $B_2 = \{n, s\}$.



The degeneracy conditions imply

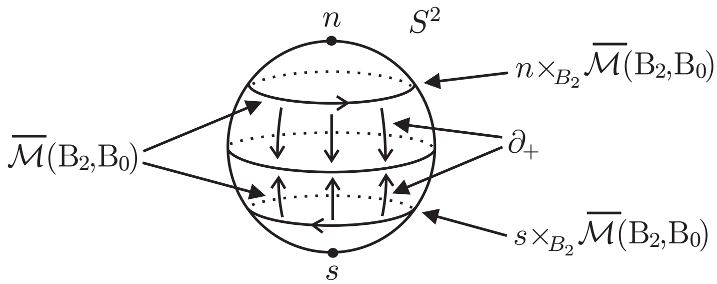
$$S_0^\infty(B_2)/D_0^\infty(B_2) \approx < n, s > \approx \mathbb{Z} \oplus \mathbb{Z},$$

and $S_p^\infty(B_2)/D_p^\infty(B_2) = 0$ for $p > 0$.

$$\begin{array}{ccccccc}
< n, s > & \xrightarrow{\partial_0} & 0 & & & & \\
\oplus & \searrow^{\partial_1} \ \searrow^{\partial_2} & \oplus & & & & \\
0 & \xrightarrow{\partial_0} & 0 & \xrightarrow{\partial_0} & 0 & & \\
\oplus & \searrow^{\partial_1} & \oplus & \searrow^{\partial_1} & \oplus & & \\
S_2^\infty(B_0)/D_2^\infty(B_0) & \xrightarrow{\partial_0} & S_1^\infty(B_0)/D_1^\infty(B_0) & \xrightarrow{\partial_0} & S_0^\infty(B_0)/D_0^\infty(B_0) & \xrightarrow{\partial_0} & 0 \\
\| & & \| & & \| & & \\
C_2(f) & \xrightarrow{\partial} & C_1(f) & \xrightarrow{\partial} & C_0(f) & \xrightarrow{\partial} & 0
\end{array}$$

The group $S_k^\infty(B_0)/D_k^\infty(B_0)$ is non-trivial for all $k \leq N$, but $H_k(C_*(f), \partial) = 0$ if $k > 2$ and $\partial_0 : S_3^\infty(B_0)/D_3^\infty(B_0) \to S_2^\infty(B_0)/D_2^\infty(B_0)$ maps onto the kernel of the boundary operator $\partial_0 : S_2^\infty(B_0)/D_2^\infty(B_0) \to S_1^\infty(B_0)/D_1^\infty(B_0)$ because the bottom row in the above diagram computes the smooth integral singular homology of $B_0 \approx S^1$.

The moduli space $\overline{\mathcal{M}}(B_2, B_0)$ is a disjoint union of two copies of S^1 with opposite orientations. This moduli space can be viewed as a subset of the manifold S^2 since $\overline{\mathcal{M}}(B_2, B_0) = \mathcal{M}(B_2, B_0)$.



There is an orientation reversing map $\alpha : n \times_n \overline{\mathcal{M}}(B_2, B_0) \to s \times_s \overline{\mathcal{M}}(B_2, B_0)$ such that $\partial_2(n) \circ \alpha = \partial_2(s)$. Since $\partial_0(\partial_2(n)) = \partial_0(\partial_2(s)) = 0$, the degeneracy conditions imply that

$$\partial_2(n + s) = \partial_2(n) + \partial_2(s) = 0 \in S_1(B_0)/D_1(B_0).$$

They also imply that ∂_2 maps either n or s onto a representative of the generator of

$$\frac{\ker \partial_0 : S_1^\infty(B_0)/D_1^\infty(B_0) \to S_0^\infty(B_0)/D_0^\infty(B_0)}{\text{im } \partial_0 : S_2^\infty(B_0)/D_2^\infty(B_0) \to S_1^\infty(B_0)/D_1^\infty(B_0)} \approx H_1(S^1; \mathbb{Z}) \approx \mathbb{Z}$$

depending on the orientation chosen for B_0. Therefore,

$$H_k(C_*(f), \partial) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

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Examples with fibered products

Fibered products of simplicial complexes

Let $f : [0,1] \to [0,1] \times [-1,1]$ be given by

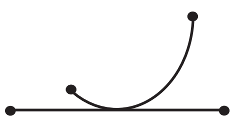
$$f(t) = \begin{cases} (t, e^{-1/t^2} \sin(\pi/t)) & \text{if } t \neq 0 \\ (0,0) & \text{if } t = 0 \end{cases}$$

and $g : [0,1] \times [0,1] \to [0,1] \times [-1,1]$ be given by $g(x,y) = (x,0)$. Then f and g are maps between finite simplicial complexes whose fibered product $[0,1] \times_{(f,g)} [0,1] \times [0,1] =$

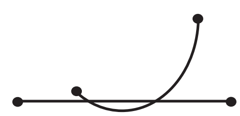
$$\{(t,t,0) \in [0,1] \times [0,1] \times [0,1] |\ t = 0, 1, 1/2, 1/3, \ldots\}$$

is not a finite simplicial complex.

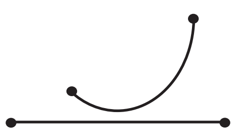
Perturbations and fibered products



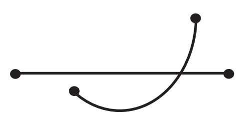
A non-transverse point of intersection



Two transverse points of intersections



No intersection points



One transverse point of intersection

If $f : P_1 \to B$ and $g : P_2 \to B$ do not meet transversally, and we perturb f to $\tilde{f} : P_1 \to B$ so that $\tilde{f}$ and g do meet transversally, then the fibered product

$$P_1 \times_{(\tilde{f},g)} P_2$$

might depend on the perturbation.

Triangulations and fibered products

Having triangulations on two spaces does not immediately induce a triangulation on the fibered product. In fact, there are simple diagrams of polyhedra and piecewise linear maps for which the diagram is not triangulable:

$$R \stackrel{g}{\leftarrow} P \stackrel{f}{\rightarrow} Q$$

There may not exist triangulations of P, Q, and R with respect to which both f and g are simplicial. [J.L. Bryant, *Triangulation and general position of PL diagrams*, Top. App. **34** (1990), 211-233]

The Banyaga-Hurtubise approach

1. Work in the category of compact smooth manifolds with corners instead of the category of finite simplicial complexes.

2. They prove that all of the relevant fibered products are compact smooth manifolds with corners.

3. They prove that it is not necessary to perturb the beginning and endpoint maps to achieve transversality. So, they don't have to worry about the fibered products changing under perturbations.

4. They don't have to deal with any issues involving triangulations because their approach allows singular chains whose domains are spaces more general than a simplex.