MULTICOMPLEXES AND SPECTRAL SEQUENCES

DAVID E. HURTUBISE

Abstract. In this note we present some algebraic examples of multicomplexes whose differentials differ from those in the spectral sequences associated to the multicomplexes. The motivation for constructing examples showing the algebraic distinction between a multicomplex and its associated spectral sequence comes from the author's work on Morse-Bott homology with A. Banyaga [1].

1. Introduction

Let R be a principal ideal domain. A first quadrant **multicomplex** X is a bigraded R-module $\{X_{p,q}\}_{p,q\in\mathbb{Z}_+}$ with differentials

$$\mathsf{d}_i : X_{p,q} \to X_{p-i,q+i-1} \quad \text{for all } i = 0, 1, \ldots$$

that satisfy

$$\sum_{i+j=n} \mathsf{d}_i \mathsf{d}_j = 0 \quad \text{for all } n.$$

A first quadrant multicomplex such that $\mathsf{d}_i = 0$ for all $i \geq 2$ is called a **double complex** (or a **bicomplex**). For the basic properties of multicomplexes we refer the reader to [2] and [5].

An E^k first quadrant **spectral sequence** is a sequence of bigraded R-modules $\{E^r_{s,t}\}_{s,t\in\mathbb{Z}_+}$ with differentials

$$d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$$

such that for all $r \geq k$ there is a given isomorphism $H(E^r) \approx E^{r+1}$ (see Section 2). Every first quadrant multicomplex determines an E^0 first quadrant spectral sequence. However, not every first quadrant spectral sequence comes from a multicomplex.

Moreover, the differentials d^r in the spectral sequence associated to a first quadrant multicomplex are in general different from the homomorphisms induced by the differentials d_i in the multicomplex. (Note that using the term "differential" to describe the homomorphisms d_i in a multicomplex is misleading since there is no guarantee that $(\mathsf{d}_i)^2$ is zero.) The purpose of this note is to demonstrate this distinction by presenting explicit algebraic examples of multicomplexes where the differential d^r in the associated spectral sequence is different than the homomorphism induced by d_r.

2000 *Mathematics Subject Classification.* Primary: 18G40; 55T99.

The motivation for constructing examples that show the distinction between a multicomplex and its associated spectral sequence comes from the author's work on Morse-Bott homology with A. Banyaga and the discovery that the Morse-Bott-Smale chain complex is in fact a multicomplex. For more details see the introduction to [1].

2. The spectral sequence associated to a filtered chain complex

In this section we clarify the meaning (and the bigrading) of the isomorphism $H(E^r) \approx E^{r+1}$, and we recall the definition of the differentials $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ in an E^k spectral sequence coming from a filtered chain complex. This section follows Chapter 9 of [6].

An E^k spectral sequence consists of a sequence of bigraded modules $\{E^r_{s,t}\}$ over a principal ideal domain R for $r \geq k$, with differentials $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ that satisfy $(d^r)^2 = 0$. If we define

$$
\begin{aligned}
\bar{Z}^r_{s,t} &= \ker(d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}) \\
\bar{B}^r_{s,t} &= \operatorname{im}(d^r : E^r_{s+r,t-r+1} \to E^r_{s,t})
\end{aligned}
$$

then $\bar{B}^r_{s,t} \subseteq \bar{Z}^r_{s,t}$, and by definition there is a given isomorphism $E^{r+1}_{s,t} \approx \bar{Z}^r_{s,t}/\bar{B}^r_{s,t}$.

Let (C_*, ∂) be a filtered chain complex that is bounded below by $s = 0$. That is, suppose that we have a filtration

$$
F_0C_* \subset \cdots \subset F_{s-1}C_* \subset F_sC_* \subset F_{s+1}C_* \subset \cdots
$$

where F_sC_* is a chain subcomplex of C_* for all s, i.e. $\partial(F_sC_{s+t}) \subseteq F_sC_{s+t-1}$ for all t. The grading s is called the **filtered degree**, the grading t is called the **complementary degree**, and the sum $s+t$ is called the **total degree**. The filtration is said to be **convergent** if $\cap_s F_sC_* = 0$ and $\cup_s F_sC_* = C_*$. Define

$$
\begin{aligned}
Z^r_{s,t} &= \{c \in F_sC_{s+t} |\ \partial c \in F_{s-r}C_{s+t-1}\} \\
Z^\infty_{s,t} &= \{c \in F_sC_{s+t} |\ \partial c = 0\}.
\end{aligned}
$$

The bigraded R-modules in the spectral sequence associated to the filtration are defined to be

$$
\begin{aligned}
E^r_{s,t} &= Z^r_{s,t} \big/ \big(Z^{r-1}_{s-1,t+1} + \partial Z^{r-1}_{s+r-1,t-r+2}\big) \\
E^\infty_{s,t} &= Z^\infty_{s,t} \big/ \big(Z^\infty_{s-1,t+1} + (\partial C_{s+t+1} \cap F_sC_{s+t})\big)\ ,
\end{aligned}
$$

where $A+B$ denotes the free abelian group generated by the elements of A and B, and the differential $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ is defined by the following diagram.

$$\begin{array}{ccc} Z^r_{s,t} & \xrightarrow{\partial} & Z^r_{s-r,t+r-1} \\ \downarrow & & \downarrow \\ Z^r_{s,t} \big/ \big(Z^{r-1}_{s-1,t+1} + \partial Z^{r-1}_{s+r-1,t-r+2}\big) & \xrightarrow{d^r} & Z^r_{s-r,t+r-1} \big/ \big(Z^{r-1}_{s-r-1,t+r} + \partial Z^{r-1}_{s-1,t+1}\big) \end{array}$$

The R-module $E^r_{s,t}$ is isomorphic to $\bar{Z}^{r-1}_{s,t}/\bar{B}^{r-1}_{s,t}$ via an isomorphism given by the Noether Isomorphism Theorem.

For a proof of the following theorem see Section 9.1 of [6].

Theorem 1. *If the filtration on the chain complex* (C_*, ∂) *is convergent and bounded below, then the above spectral sequence converges to the bigraded* R*-module* $GH_*(C_*, \partial)$ *associated to the filtration* $F_sH_*(C_*, \partial) \equiv im[H_*(F_sC_*, \partial) \to H_*(C_*, \partial)]$*. That is,*

$$E^\infty_{s,t} \approx \bigcap_r Z^r_{s,t} \Big/ \bigcup_r \big(Z^{r-1}_{s-1,t+1} + \partial Z^{r-1}_{s+r-1,t-r+2}\big) \approx GH_*(C_*, \partial)_{s,t}$$

where $GH_*(C_*, \partial)_{s,t} \equiv F_sH_{s+t}(C_*, \partial)/F_{s-1}H_{s+t}(C_*, \partial)$.

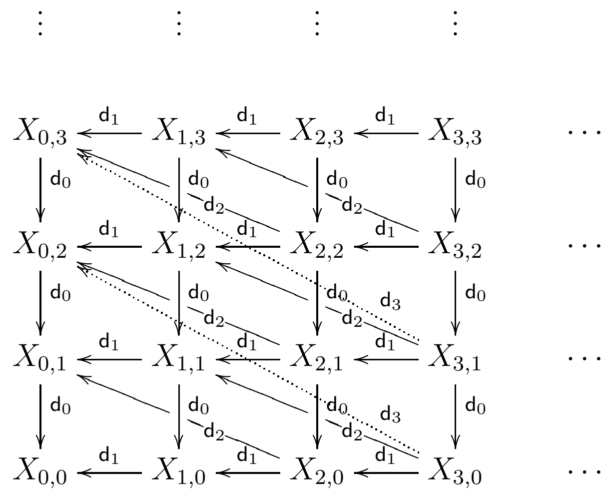
3. The spectral sequence associated to a multicomplex

A first quadrant multicomplex $(\{X_{p,q}\}_{p,q\in\mathbb{Z}_+}, \{\mathsf{d}_i\}_{i\in\mathbb{Z}_+})$ can be **assembled** to form a filtered chain complex $((CX)_*, \partial)$ by summing along the diagonals. That is, suppose that we are given a bigraded R-module $\{X_{p,q}\}_{p,q\in\mathbb{Z}_+}$ and homomorphisms

$$\mathsf{d}_i : X_{p,q} \to X_{p-i,q+i-1} \quad \text{for all } i = 0, 1, \ldots$$

that satisfy

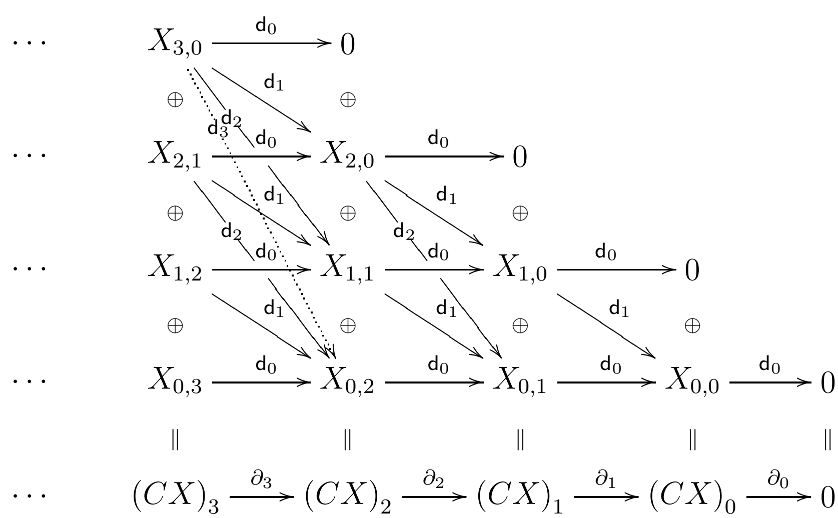
$$\sum_{i+j=n} \mathsf{d}_i \mathsf{d}_j = 0 \quad \text{for all } n.$$



If we define

$$(CX)_n \equiv \bigoplus_{p+q=n} X_{p,q}$$

and $\partial_n = \mathsf{d}_0 \oplus \cdots \oplus \mathsf{d}_n$ for all $n \in \mathbb{Z}_+$, then the above relations imply that $\partial_n \circ \partial_{n+1} = 0$.



Moreover, the chain complex $((CX)_*, \partial)$ has an obvious filtration given by

$$F_s(CX)_n \equiv \bigoplus_{\substack{p+q=n \\ p \leq s}} X_{p,q}.$$

Note that the restriction $q \leq s$ determines a second filtration on a double complex, but it does not determine a filtration on a general multicomplex.

The bigraded module associated to the above filtration is

$$G((CX)_*)_{s,t} = F_s(CX)_{s+t}/F_{s-1}(CX)_{s+t} \approx X_{s,t}$$

for all $s, t \in \mathbb{Z}_+$, and the E^1 term of the associated spectral sequence is given by

$$E^1_{s,t} = Z^1_{s,t} \,/ \big(Z^0_{s-1,t+1} + \partial Z^0_{s,t+1}\big)$$

where

$$Z^1_{s,t} = \{c \in F_s(CX)_{s+t} |\ \partial c \in F_{s-1}(CX)_{s+t-1}\}$$

and

$$Z^0_{s,t} = \{c \in F_s(CX)_{s+t} |\ \partial c \in F_s(CX)_{s+t-1}\} = F_s(CX)_{s+t}.$$

The group $Z^1_{s,t}/Z^0_{s-1,t+1}$ is the group of $(s+t)$-cycles in the quotient chain complex $F_s(CX)_*/F_{s-1}(CX)_*$, and the group of $(s+t)$-boundaries in $F_s(CX)_*/F_{s-1}(CX)_*$ is

$$\partial Z^0_{s,t+1} \,/ (Z^0_{s-1,t+1} \cap \partial Z^0_{s,t+1}) \approx \big(Z^0_{s-1,t+1} + \partial Z^0_{s,t+1}\big) \,/ Z^0_{s-1,t+1}$$

where the isomorphism is given by the Noether Isomorphism Theorem. Therefore,

$$\begin{aligned} E^1_{s,t} &= Z^1_{s,t} \,/ \big(Z^0_{s-1,t+1} + \partial Z^0_{s,t+1}\big) \\ &\approx \frac{Z^1_{s,t}/Z^0_{s-1,t+1}}{\big(Z^0_{s-1,t+1} + \partial Z^0_{s,t+1}\big) \,/ Z^0_{s-1,t+1}} \\ &\approx \frac{Z^1_{s,t}/Z^0_{s-1,t+1}}{\partial Z^0_{s,t+1} \,/ (Z^0_{s-1,t+1} \cap \partial Z^0_{s,t+1})} \end{aligned}$$

and we see that $E^1_{s,t} \approx H_{s+t}(X_{s,*}, \mathsf{d}_0)$ where $(X_{s,*}, \mathsf{d}_0)$ denotes the chain complex

$$\cdots \xrightarrow{\mathsf{d}_0} X_{s,3} \xrightarrow{\mathsf{d}_0} X_{s,2} \xrightarrow{\mathsf{d}_0} X_{s,1} \xrightarrow{\mathsf{d}_0} X_{s,0} \xrightarrow{\mathsf{d}_0} 0.$$

The differential d^1 on the E^1 term of the spectral sequence is defined by the diagram

$$\begin{array}{ccc} Z^1_{s,t} & \xrightarrow{\quad\partial\quad} & Z^1_{s-1,t} \\ \big\downarrow & & \big\downarrow \\ Z^1_{s,t} \,/ \big(Z^0_{s-1,t+1} + \partial Z^0_{s,t+1}\big) & \xrightarrow{d^1} & Z^1_{s-1,t} \,/ \big(Z^0_{s-2,t+1} + \partial Z^0_{s-1,t+1}\big) \end{array}$$

and it is natural to ask whether or not there is a connection between the differential $d^1 : E^1_{s,t} \to E^1_{s-1,t}$ in the spectral sequence and the homomorphism $\mathsf{d}_1 : X_{s,t} \to X_{s-1,t}$ in the multicomplex.

It is an easy exercise to show that the relations

$$\begin{aligned} \mathsf{d}_0\mathsf{d}_1 + \mathsf{d}_1\mathsf{d}_0 &= 0 \\ \mathsf{d}_0\mathsf{d}_2 + \mathsf{d}_1\mathsf{d}_1 + \mathsf{d}_2\mathsf{d}_0 &= 0 \end{aligned}$$

imply that the homomorphism d_1 induces a differential $\bar{\mathsf{d}}_1 : E^1_{s,t} \to E^1_{s-1,t}$, i.e. $(\bar{\mathsf{d}}_1)^2 = 0$. Moreover, one can show that the differential $\bar{\mathsf{d}}_1$ coincides with the differential $d^1 : E^1_{s,t} \to E^1_{s-1,t}$. This is a standard fact for double complexes, and the proof for double complexes carries over to multicomplexes (see for instance Section 14 of [3], or Section 3.2.1 of [4]). Thus, we have the following.

Theorem 2. *Let* $(\{X_{p,q}\}_{p,q\in\mathbb{Z}_+}, \{\mathsf{d}_i\}_{i\in\mathbb{Z}_+})$ *be a first quadrant multicomplex and* $((CX)_*, \partial)$ *the associated assembled chain complex. Then the* E^1 *term of the spectral sequence associated to the filtration of* $(CX)_*$ *determined by the restriction* $p \leq s$ *is given by* $E^1_{s,t} \approx H_{s+t}(X_{s,*}, \mathsf{d}_0)$ *where* $(X_{s,*}, \mathsf{d}_0)$ *denotes the following chain complex.*

$$\cdots \xrightarrow{\mathsf{d}_0} X_{s,3} \xrightarrow{\mathsf{d}_0} X_{s,2} \xrightarrow{\mathsf{d}_0} X_{s,1} \xrightarrow{\mathsf{d}_0} X_{s,0} \xrightarrow{\mathsf{d}_0} 0$$

Moreover, the d^1 *differential on the* E^1 *term of the spectral sequence is induced from the homomorphism* d_1 *in the multicomplex.*

4. Multicomplexes where $d^r \neq \mathbf{d}_r$

Theorem 2 should sound familiar to anyone acquainted with double complexes. However, the examples in this section show that Theorem 2 does **not** generalize to the higher differentials in the spectral sequence associated to a multicomplex. In fact, the pattern suggested by Theorem 2 breaks down when $r = 2$. That is, the differential d^r in the spectral sequence is **not** necessarily induced from the homomorphism d_r when $r \geq 2$. To paraphrase Section 11 of [2], when $r \geq 2$ the differential d^r is induced from d_r only on those classes which contain elements x such that $\mathsf{d}_i(x) = 0$ for all $i < r$ "which rarely happens".

Example 1 (A double complex with $d^2 \neq 0$)**.**

It is well known that the spectral sequence associated to a double complex does not necessarily degenerate at E^2. That is, there is no guarantee that $d^r = 0$ for $r \geq 2$. This first example is a small algebraic example that demonstrates this phenomena.

Consider the following first quadrant double complex

$$\begin{array}{ccccc}
0 & \xleftarrow{\mathsf{d}_1} & 0 & \xleftarrow{\mathsf{d}_1} & 0 \\
\downarrow{\scriptstyle \mathsf{d}_0} & & \downarrow{\scriptstyle \mathsf{d}_0} & & \downarrow{\scriptstyle \mathsf{d}_0} \\
< x_{0,1} > & \xleftarrow{\mathsf{d}_1} & < x_{1,1} > & \xleftarrow{\mathsf{d}_1} & 0 \\
\downarrow{\scriptstyle \mathsf{d}_0} & & \downarrow{\scriptstyle \mathsf{d}_0} & & \downarrow{\scriptstyle \mathsf{d}_0} \\
0 & \xleftarrow{\mathsf{d}_1} & < x_{1,0} > & \xleftarrow{\mathsf{d}_1} & < x_{2,0} >
\end{array}$$

where $< x_{p,q} >$ denotes the free abelian group generated by $x_{p,q}$, the groups $X_{p,q} = 0$ for $p + q > 2$, and the homomorphisms d_0 and d_1 satisfy the following: $\mathsf{d}_0(x_{1,1}) = x_{1,0}$, $\mathsf{d}_1(x_{1,1}) = x_{0,1}$, and $\mathsf{d}_1(x_{2,0}) = x_{1,0}$. The conditions $(\mathsf{d}_0)^2 = (\mathsf{d}_1)^2 = 0$ and $\mathsf{d}_0\mathsf{d}_1 + \mathsf{d}_1\mathsf{d}_0 = 0$ are satisfied trivially, and the assembled chain complex associated to this double complex is as follows.

$$
\begin{array}{cccccccccc}
\cdots & 0 & \xrightarrow{\mathsf{d}_0} & 0 & & & & & & \\
 & & \searrow^{\mathsf{d}_1} & & & & & & & \\
 & \oplus & & \oplus & & & & & & \\
\cdots & 0 & \xrightarrow{\mathsf{d}_0} & < x_{2,0} > & \xrightarrow{\mathsf{d}_0} & 0 & & & & \\
 & & \searrow^{\mathsf{d}_1} & & \searrow^{\mathsf{d}_1} & & & & & \\
 & \oplus & & \oplus & & \oplus & & & & \\
\cdots & 0 & \xrightarrow{\mathsf{d}_0} & < x_{1,1} > & \xrightarrow{\mathsf{d}_0} & < x_{1,0} > & \xrightarrow{\mathsf{d}_0} & 0 & & \\
 & & \searrow^{\mathsf{d}_1} & & \searrow^{\mathsf{d}_1} & & \searrow^{\mathsf{d}_1} & & & \\
 & \oplus & & \oplus & & \oplus & & \oplus & & \\
\cdots & 0 & \xrightarrow{\mathsf{d}_0} & 0 & \xrightarrow{\mathsf{d}_0} & < x_{0,1} > & \xrightarrow{\mathsf{d}_0} & 0 & \xrightarrow{\mathsf{d}_0} & 0 \\
 & \| & & \| & & \| & & \| & & \| \\
\cdots & 0 & \xrightarrow{\partial_3} & (CX)_2 & \xrightarrow{\partial_2} & (CX)_1 & \xrightarrow{\partial_1} & 0 & \xrightarrow{\partial_0} & 0
\end{array}
$$

The homology $H_n((CX)_*, \partial)$ of the assembled chain complex is trivial for all $n \in \mathbb{Z}_+$ because the kernel of ∂_2 is trivial and both $x_{0,1}$ and $x_{1,0}$ are in the image of $\partial_2 = \mathsf{d}_0 + \mathsf{d}_1$:

$$
\begin{aligned}
\partial_2(x_{2,0}) &= x_{1,0} \\
\partial_2(x_{1,1} - x_{2,0}) &= x_{0,1}.
\end{aligned}
$$

However, the E^1 term of the associated spectral sequence is

$$
\begin{array}{ccccc}
0 & \xleftarrow{\mathsf{d}_1} & 0 & \xleftarrow{\mathsf{d}_1} & 0 \\
 & & & & \\
< x_{0,1} > & \xleftarrow{\mathsf{d}_1} & 0 & \xleftarrow{\mathsf{d}_1} & 0 \\
 & & & & \\
0 & \xleftarrow{\mathsf{d}_1} & 0 & \xleftarrow{\mathsf{d}_1} & < x_{2,0} >
\end{array}
$$

where $E^1_{s,t} = 0$ for all $s + t > 2$, and the E^2 term is isomorphic to the E^1 term. Since $H_n((CX)_*, \partial) = 0$ for all $n \in \mathbb{Z}_+$, Theorem 1 implies that the differential d^2 in the spectral sequence must be nonzero, even though the homomorphism d_2 in the multicomplex is zero, i.e. $d^2 \neq 0$ is **not** induced from $\mathsf{d}_2 = 0$.

To compute the differential $d^2 : E^2_{2,0} \to E^2_{0,1}$ we consider the following diagram

$$
\begin{array}{ccc}
Z^2_{2,0} & \xrightarrow{\partial} & Z^2_{0,1} \\
\downarrow & & \downarrow \\
Z^2_{2,0} \big/ \big(Z^1_{1,1} + \partial Z^1_{3,0}\big) & \xrightarrow{d^2} & Z^2_{0,1} \big/ \big(Z^1_{-1,2} + \partial Z^1_{1,1}\big)
\end{array}
$$

where

$$
Z^2_{2,0} = < x_{1,1} - x_{2,0} >, \quad Z^2_{0,1} = < x_{0,1} >
$$

and $Z^1_{1,1} = Z^1_{3,0} = Z^1_{-1,2} = 0$. Since $\partial_2(x_{1,1} - x_{2,0}) = x_{0,1}$, we see that the E^3 term of the spectral sequence is trivial, and we have verified Theorem 1 in this example.

There is one more subtle point to note in this example: although $E^1_{2,0}$ and $E^2_{2,0}$ are isomorphic, they have different generators. That is,

$$E^1_{2,0} = Z^1_{2,0} \big/ \big(Z^0_{1,1} + \partial Z^0_{2,1}\big) \approx < x_{2,0} >$$

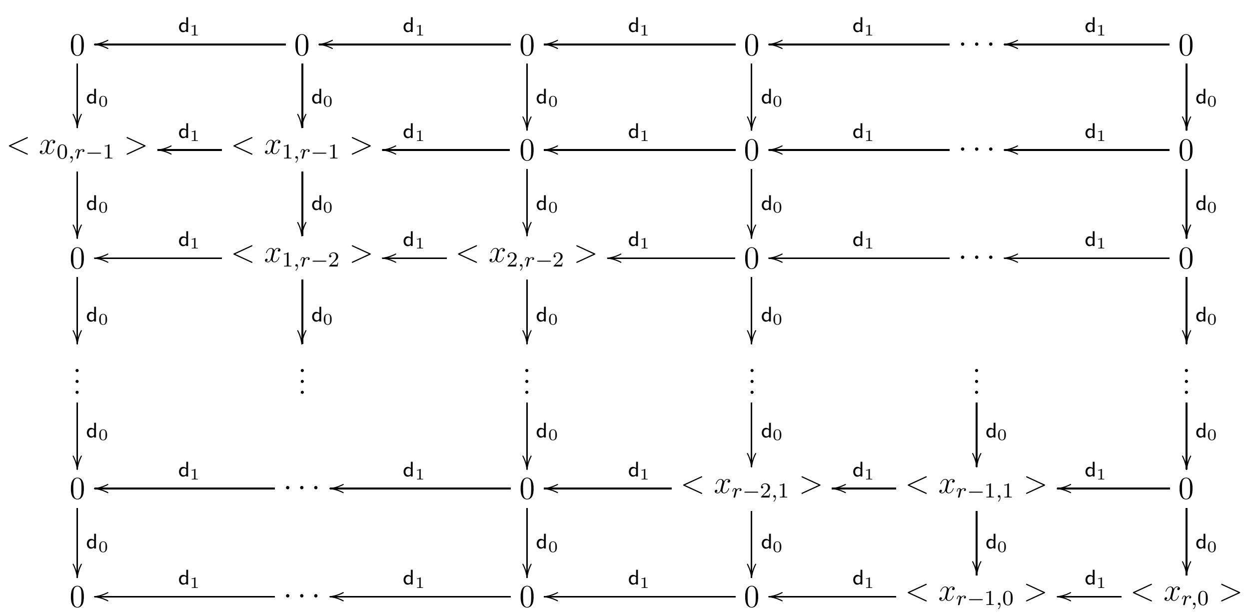
whereas

$$E^2_{2,0} = Z^2_{2,0} \big/ \big(Z^1_{1,1} + \partial Z^1_{3,0}\big) \approx < x_{1,1} - x_{2,0} > .$$

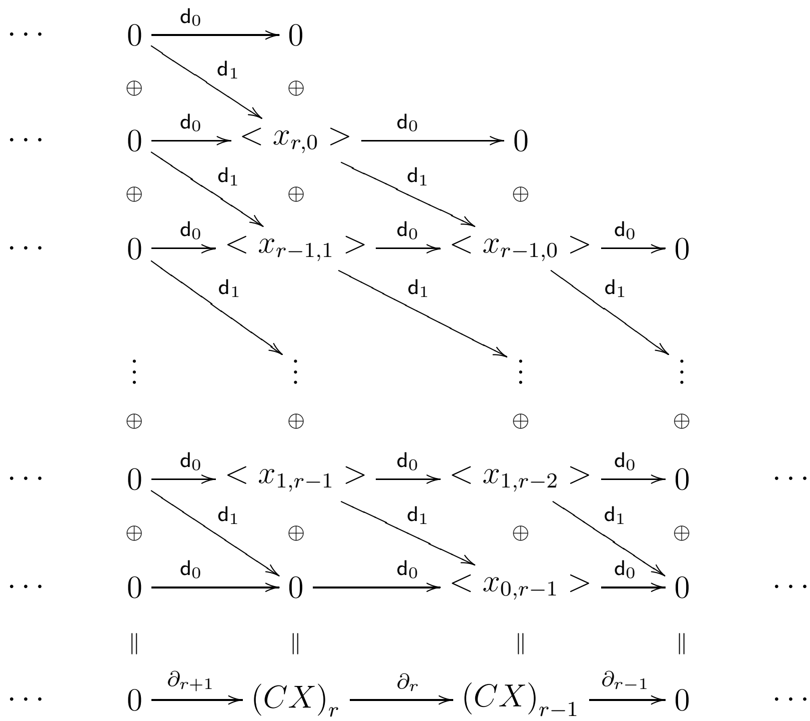
This is consistent with the definition of a spectral sequence which states that "there is a given isomorphism $H(E^r) \approx E^{r+1}$" [6].

Example 2 (A double complex with some $d^r \neq 0$ for r arbitrarily large)**.**

The preceeding example can be generalized to produce a double complex such that a differential d^r in the associated spectral sequence is nonzero for r arbitrarily large. To see this pick any $r \in \mathbb{Z}_+$ with $r \geq 2$, and consider the following first quadrant double complex



where the groups $X_{p,q} = 0$ for $p + q > r$ and the homomorphisms d_0 and d_1 satisfy the following for $p + q = r$: $\mathsf{d}_0(x_{p,q}) = x_{p,q-1}$ and $\mathsf{d}_1(x_{p,q}) = x_{p-1,q}$. The conditions $(\mathsf{d}_0)^2 = (\mathsf{d}_1)^2 = 0$ and $\mathsf{d}_0\mathsf{d}_1 + \mathsf{d}_1\mathsf{d}_0 = 0$ are satisfied trivially, and the assembled chain complex associated to this double complex is as follows.



As in the previous example, the homology $H_n((CX)_*, \partial)$ of the assembled chain complex is trivial for all $n \in \mathbb{Z}_+$ because the kernel of ∂_r is trivial and all the generators $x_{0,r-1}, x_{1,r-2}, \ldots, x_{r-1,0}$ of $(CX)_{r-1}$ are in the image of ∂_r. The E^1 term of the associated spectral sequence has $E^1_{0,r-1} = < x_{0,r-1} >$, $E^1_{r,0} = < x_{r,0} >$, and $E^1_{s,t} = 0$ for all other values of s and t. Moreover, $E^1 \approx E^2 \approx \cdots \approx E^{r-1}$. Once again, Theorem 1 implies that the differential d^r must be nonzero (even though $\mathsf{d}_r = 0$), and the diagram

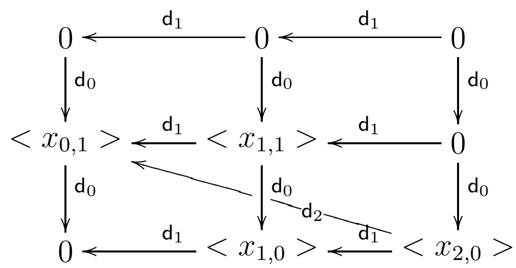
$$\begin{array}{ccc} Z^r_{r,0} & \xrightarrow{\quad\partial\quad} & Z^r_{0,r-1} \\ \downarrow & & \downarrow \\ Z^r_{r,0}\big/\big(Z^{r-1}_{r-1,1} + \partial Z^{r-1}_{2r-1,-r+2}\big) & \xrightarrow{d^r} & Z^r_{0,r-1}\big/\big(Z^{r-1}_{-1,r} + \partial Z^{r-1}_{r-1,1}\big) \end{array}$$

can be used to show that d^r is surjective. (Note that $Z^r_{r,0}$ is generated by $x_{1,r-1} - x_{2,r-2} + \cdots + (-1)^{r-1} x_{r,0}$.)

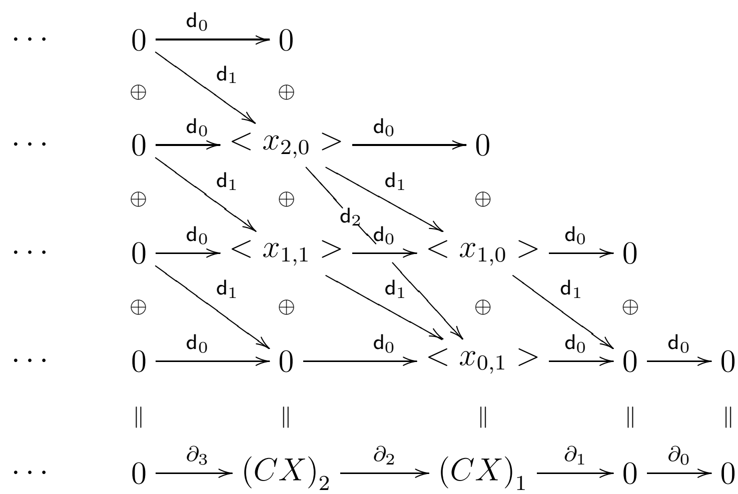
Example 3 (Multicomplexes with $\mathsf{d}_r \neq 0$ and $d^i = 0$ for all $i \geq 2$)**.**

The preceeding examples show that the spectral sequence associated to a multicomplex with $\mathsf{d}_r = 0$ for all $r \geq 2$ may not degenerate at E^2 (or even E^r where r is arbitrarily large). These examples can be modified to show that there exist multicomplexes where $\mathsf{d}_r \neq 0$ for r arbitrarily large but the associated spectral sequences degenerate at E^2.

We begin with a multicomplex where $\mathsf{d}_2 \neq 0$ but its associated spectral sequence degenerates at E^2. Consider the following first quadrant multicomplex



where the groups $X_{p,q} = 0$ for $p+q > 2$, the homomorphisms d_0 and d_1 are the same as in Example 1, and $\mathsf{d}_2(x_{2,0}) = x_{0,1}$. The homomorphisms $\mathsf{d}_i : X_{p,q} \to X_{p-i,q+i-1}$ satisfy $\sum_{i+j=n} \mathsf{d}_i \mathsf{d}_j = 0$ for all n trivially, and the assembled chain complex associated to this multicomplex is as follows.



Referring back to Example 1 we see that the diagram

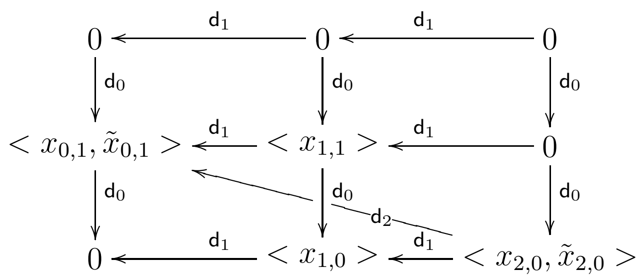
$$\begin{array}{ccc}
Z^r_{2,0} & \xrightarrow{\quad\partial\quad} & Z^r_{2-r,r-1} \\
\downarrow & & \downarrow \\
Z^r_{2,0} \big/ \big(Z^{r-1}_{1,1} + \partial Z^{r-1}_{r+1,-r+2} \big) & \xrightarrow{d^r} & Z^r_{2-r,r-1} \big/ \big(Z^{r-1}_{1-r,r} + \partial Z^{r-1}_{1,1} \big)
\end{array}$$

shows that $d^r = 0$ for all $r \geq 2$ since $\partial_2(x_{1,1} - x_{2,0}) = 0$, and hence the associated spectral sequence degenerates at E^2.

To generalize this example to produce a multicomplex where $\mathsf{d}_r \neq 0$ for r arbitrarily large but the associated spectral sequence still degenerates at E^2 we start with the double complex from Example 2 and add a single homomorphism d_r defined by $\mathsf{d}_r(x_{r,0}) = (-1)^r x_{0,r-1}$. Further details are left to the reader.

Example 4 (A multicomplex with $d^2 \neq 0$, $\mathsf{d}_2 \neq 0$ and $d^2 \neq \mathsf{d}_2$).

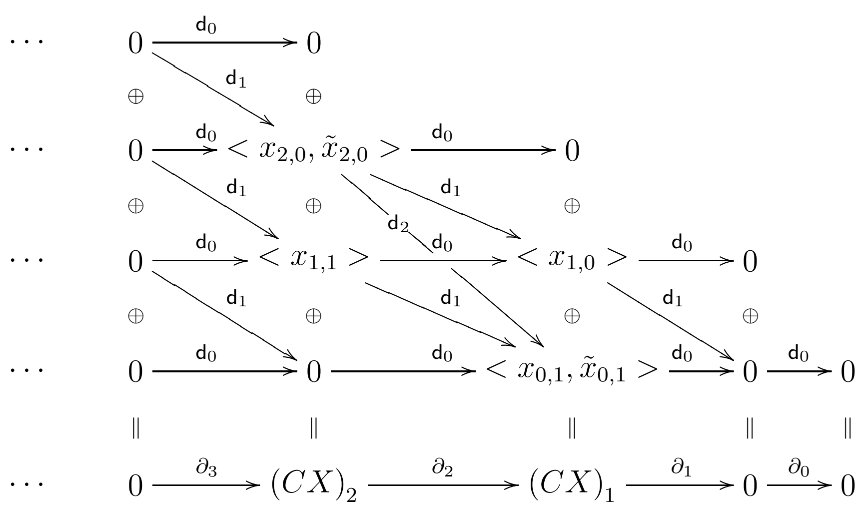
Consider the following first quadrant multicomplex



where the groups $X_{p,q} = 0$ for $p + q > 2$, and the homomorphisms d_i for $i = 0, 1, 2$ satisfy the following.

$$\begin{array}{rcr} \mathsf{d}_0(x_{1,1}) & = & x_{1,0} \\ \mathsf{d}_1(x_{1,1}) & = & x_{0,1} \\ \mathsf{d}_1(x_{2,0}) & = & x_{1,0} \\ \mathsf{d}_1(\tilde{x}_{2,0}) & = & 0 \\ \mathsf{d}_2(\tilde{x}_{2,0}) & = & \tilde{x}_{0,1} \\ \mathsf{d}_2(x_{2,0}) & = & 0 \end{array}$$

The homomorphisms $\mathsf{d}_i : X_{p,q} \to X_{p-i,q+i-1}$ satisfy $\sum_{i+j=n} \mathsf{d}_i \mathsf{d}_j = 0$ for all n trivially, and the assembled chain complex associated to this multicomplex is as follows.

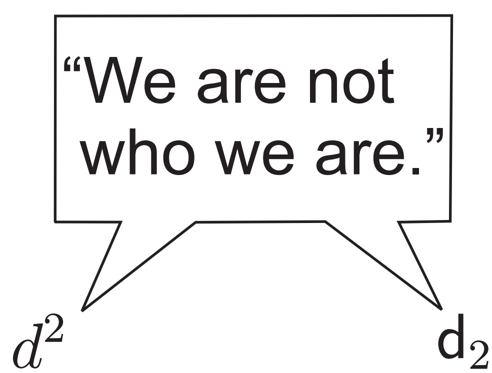


The homology $H_n((CX)_*, \partial)$ of the assembled chain complex is trivial for all $n \in \mathbb{Z}_+$, the E^1 term of the associated spectral sequence is

$$\begin{array}{ccccc} 0 & \xleftarrow{\mathsf{d}_1} & 0 & \xleftarrow{\mathsf{d}_1} & 0 \\ & & & & \\ < x_{0,1}, \tilde{x}_{0,1} > & \xleftarrow{\mathsf{d}_1} & 0 & \xleftarrow{\mathsf{d}_1} & 0 \\ & & & & \\ 0 & \xleftarrow{\mathsf{d}_1} & 0 & \xleftarrow{\mathsf{d}_1} & < x_{2,0}, \tilde{x}_{2,0} > \end{array}$$

where $E^1_{s,t} = 0$ for all $s + t > 2$, and the E^2 term is isomorphic to the E^1 term. The image of the homomorphism induced by d_2 does not include the class determined by $x_{0,1}$. However, the differential d^2 in the spectral sequence is onto. Therefore, d^2 is not the same as the homomorphism induced by d_2.

Note that additional examples can be constructed where the homology is nontrivial by adding more generators. Examples 1, 2, and 4 were constructed to have trivial homology in order to make it easy to see that d^r is surjective. Also, it should be clear at this point how to construct examples where d^r is not induced from d_r for several different values of r: simply combine the above examples using more (disjoint) generators.



Acknowledgments: I would like to thank Jim Stasheff and other members of the University of Pennsylvania's Deformation Theory Seminar, in particular Tom Hunter and Ron Umble, for introducing me to multicomplexes and pointing out that the Morse-Bott-Smale chain complex is in fact a multicomplex.

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Department of Mathematics and Statistics, Penn State Altoona, Altoona, PA 16601-3760
E-mail address: Hurtubise@psu.edu