SPACES OF HOLOMORPHIC MAPS FROM $\mathbb{C}P^1$ TO COMPLEX GRASSMANN MANIFOLDS

DAVID E. HURTUBISE

1. Introduction

In this note we provide a detailed proof of a "well-known folk theorem." This theorem has been used by many authors who study the topology of spaces of holomorphic maps [1] [7] [5]. The theorem gives a description of the space of holomorphic maps from $\mathbb{C}P^1$ to the complex Grassmann manifold $G_{n,n+k}(\mathbb{C})$ in terms of equivalence classes of λ-matrices $M_{n,n+k}(\mathbb{C}[z])$, i.e. $n \times (n+k)$ matrices with entries in the polynomial ring $\mathbb{C}[z]$. The equivalence relation is given by the action of the topological group $GL_n(\mathbb{C}[z])$ consisting of those $n \times n$ λ-matrices whose determinant is a non-zero constant. This group acts on the space of $n \times (n+k)$ λ-matrices by matrix multiplication on the left.

We will show that the action

$$GL_n(\mathbb{C}[z]) \times M_{n,n+k}(\mathbb{C}[z]) \to M_{n,n+k}(\mathbb{C}[z])$$

restricts to an action

$$GL_n(\mathbb{C}[z]) \times P_{n,n+k}(\mathbb{C}[z]) \to P_{n,n+k}(\mathbb{C}[z])$$

where $P_{n,n+k}(\mathbb{C}[z])$ is the space of polynomial maps from $\mathbb{C}$ to the Stiefel manifold $V_{n,n+k}(\mathbb{C})$. The quotient space is in bijective correspondence with the space of holomorphic maps $f : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$.

$$\mathrm{Hol}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \longleftrightarrow P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$$

The space of holomorphic maps $f : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ of degree d corresponds to the subspace of $P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ consisting of those matrices such that the determinants of the minors are all polynomials of degree at most d (with at least one determinant having degree d). We will show that when restricted to the space of holomorphic maps of degree d the above bijection is a homeomorphism.

We should note that the fact that a holomorphic map from $\mathbb{C}P^1$ to $G_{n,n+k}(\mathbb{C})$ is *locally* given by a matrix of polynomials follows quickly from Chow's Theorem and the GAGA principal [8] [3]. The theorem proved in this note (without reference to Chow's Theorem or the GAGA principal) improves the local result given by Chow's Theorem.

First, we show that a holomorphic map $f : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ can be represented by a single *global* matrix of polynomials. Second, we show that the compact open topology on $\mathrm{Hol}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ agrees with the quotient topology on $P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ when one restricts to elements of degree d.

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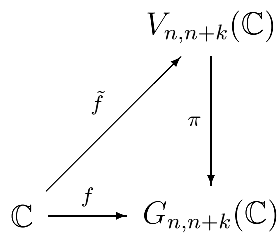
2. Holomorphic Maps and λ-Matrices

In this section we show that every holomorphic map $f : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ can be represented by a λ-matrix. That is, for every holomorphic map $f : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ there exists a polynomial map $\tilde{f} : \mathbb{C} \to V_{n,n+k}(\mathbb{C})$ such that the following diagram commutes:



where $\mathbb{C}P^1 = \mathbb{C} \cup \infty$.

Let $D^+(z_0) = \{[z_0 : z_1] \in \mathbb{C}P^1 | z_0 \neq 0\}$ and $D^+(z_1) = \{[z_0 : z_1] \in \mathbb{C}P^1 | z_1 \neq 0\}$.

$$\mathbb{C}P^1 = D^+(z_0) \cup D^+(z_1)$$

On $D^+(z_0)$ we have the chart $[z_0 : z_1] \mapsto z_1/z_0$ and on $D^+(z_1)$ we have $[z_0 : z_1] \mapsto z_0/z_1$. In terms of affine coordinates $z = z_1/z_0 \in \mathbb{C}$.

Lemma 1. *Let $\gamma_1^* \to \mathbb{C}P^1$ be the tautological holomorphic line bundle. Every holomorphic section s of the m-fold tensor product bundle $\gamma_1^{*\otimes m} \to \mathbb{C}P^1$ is a polynomial of degree $\leq m$ in the holomorphic chart on $D^+(z_0)$.*

Proof:

The transition function for $\gamma_1^{*\otimes m}$ from $D^+(z_0)$ to $D^+(z_1)$ is multiplication by $(z_0/z_1)^m$. If we let $z = z_1/z_0$ and identify $D^+(z_0)$ with $\mathbb{C}$ then since s is holomorphic we have,

$$s|_{D^+(z_0)} = \sum_{k \geq 0} a_k z^k$$

and

$$s|_{D^+(z_1)} = \sum_{k \geq 0} b_k z^{-k}.$$

On $D^+(z_0) \cap D^+(z_1)$ we have

$$(z_0/z_1)^m s|_{D^+(z_0)} = s|_{D^+(z_1)}$$

and hence

$$z^{-m} \sum_{k \geq 0} a_k z^k = \sum_{k \geq 0} b_k z^{-k}.$$

Thus,

$$\sum_{k \geq 0} a_k z^k = \sum_{k \geq 0} b_k z^{m-k}$$

for all $z \in \mathbb{C}^*$ and so we must have $a_k = 0$ for $k > m$.

□

The fact that every holomorphic map $f : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ can be represented by a λ-matrix follows essentially from the above lemma and the fact that every such holomorphic map is given by the pull-backs under f of $n + k$ sections of the tautological n-plane bundle $\gamma_n^* \to G_{n,n+k}(\mathbb{C})$ which generate the fiber at every point. We give these details first for the case $n = 1$.

The tautological holomorphic line bundle $\gamma_1^* \to \mathbb{C}P^k$ can be defined as the line bundle whose total space is $\mathbb{C}P^{k+1} \backslash \{[0 : \cdots : 0 : 1]\}$ and whose projection map is $p([z_0 : \cdots : z_{k+1}]) = [z_0 : \cdots : z_k]$ (see for instance [3] p. 42). We have an atlas on $\mathbb{C}P^k$ given by the $k + 1$ open sets

$$D^+(z_j) = \{[z_0 : \cdots : z_k] | z_j \neq 0\}$$

for all $j = 0, \ldots, k$ and holomorphic charts $D^+(z_j) \to \mathbb{C}^k$

$$[z_0 : \cdots : z_k] \mapsto (z_0/z_j, \ldots, \widehat{z_j/z_j}, \ldots, z_k/z_j) \in \mathbb{C}^k$$

where the $z_j/z_j = 1$ term is omitted. These charts induce trivializations $h_j : \gamma_1^*|_{D^+(z_j)} \to D^+(z_j) \times \mathbb{C} \to \mathbb{C}$

$$h_j([z_0 : \cdots : z_k : z_{k+1}]) = z_{k+1}/z_j$$

for all $j = 0, \ldots, k$. We have $k + 1$ holomorphic sections of γ_1^* defined by

$$s_j([z_0 : \cdots : z_k]) = [z_0 : \cdots : z_k : z_j]$$

for all $j = 0, \ldots, k$.

Lemma 2. *Let $f : X \to \mathbb{C}P^k$ be a continuous map. Then for any trivialization*

$$h : f^*(\gamma_1^*)|_U \to U \times \mathbb{C} \to \mathbb{C}$$

with $x \in U \subseteq X$ we have

$$f(x) = [h(s_0^*(x)) : \cdots : h(s_k^*(x))]$$

where s_j^* is the pull-back of s_j along f for all $j = 0, \ldots, k$.

Proof:

If we write $f(x) = [f_0(x) : \cdots : f_k(x)]$, then for any $l = 0, \cdots, k$ we have

$$s_l^*(x) = (x, [f_0(x) : \cdots : f_k(x) : f_l(x)])$$

and in the pull-back of the chart $h_j : \gamma_1^*|_{D^+(z_j)} \to D^+(z_j) \times \mathbb{C} \to \mathbb{C}$ we have

$$h_j^*(s_j^*(x)) = f_l(x)/f_j(x)$$

Thus

$$f(x) = [h_j^*(s_0^*(x)) : \cdots : h_j^*(s_k^*(x))]$$

for all $j = 0, \ldots, k$. For any chart h compatible with h_j^* we have

$$[h(s_0^*(x)) : \cdots : h(s_k^*(x))] = [h_j^*(s_0^*(x)) : \cdots : h_j^*(s_k^*(x))].$$

□

Theorem 3. *Any holomorphic map $f : \mathbb{C}P^1 \to \mathbb{C}P^k$ can be written as*

$$f(z) = [f_0(z) : \cdots : f_k(z)]$$

for all $z \in \mathbb{C}$ where $f_0(z), \ldots, f_k(z)$ are polynomials.

Proof:

This follows from the preceeding Lemma and Lemma 1. In the statement of the theorem $\mathbb{C}P^1 = D^+(z_0) \cup \infty$ and $z \in D^+(z_0) = \mathbb{C}$.

□

For general $n \in \mathbb{N}$ we define the canonical n-plane bundle $\gamma_n \to G_{n,n+k}(\mathbb{C})$ to be the bundle whose total space is

$$\{(p, v) | p \in G_{n,n+k}(\mathbb{C}), v \in p\}.$$

We define the dual of this bundle

$$\gamma_n^* = \mathrm{Hom}(\gamma_n, \mathbb{C})$$

to be the tautological holomorphic n-plane bundle over $G_{n,n+k}(\mathbb{C})$. The reader can check that this definition of γ_n^* agrees with the definition given above when $n = 1$ (see for instance [11] p. 22).

There are $n + k$ canonical holomorphic sections $s_1, \ldots, s_{n+k}$ of γ_n^* defined by

$$s_j(p)[(p, v)] = j\text{th coordinate of } v \in \mathbb{C}^{n+k}$$

for all $j = 1, \ldots, n+k$. These sections generate the fiber of γ_n^* at every point of $p \in G_{n,n+k}(\mathbb{C})$.

The holomorphic coordinate charts on $G_{n,n+k}(\mathbb{C})$ are defined as follows (see for example [4] p. 193). Given an n-plane $p \in G_{n,n+k}(\mathbb{C})$ we begin by choosing any point $\tilde{p}$ in the Stiefel manifold $V_{n,n+k}(\mathbb{C})$ above

p. $\tilde{p}$ is an n-tuple of linearly independent vectors in $\mathbb{C}^{n+k}$ which we think of as an $n \times (n+k)$ matrix of complex numbers.

$$\tilde{p} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\,n+k} \\ a_{21} & a_{22} & \cdots & a_{2\,n+k} \\ \vdots & \vdots & & \vdots \\ a_{n\,1} & a_{n\,2} & \cdots & a_{n\,n+k} \end{pmatrix}$$

We have $G_{n,n+k}(\mathbb{C}) = V_{n,n+k}(\mathbb{C})/GL_n(\mathbb{C})$ where the action of $GL_n(\mathbb{C})$ is given by matrix multiplication on the left, i.e. $\tilde{p} \sim g\tilde{p}$ for all $g \in GL_n(\mathbb{C})$. Since the rows of $\tilde{p}$ are linearly independent there is some minor, say columns $I = (i_1, \ldots, i_n)$, whose determinant is non-zero. By multiplying on the left by the inverse of the minor $\tilde{p}_I$ we get a set of vectors which span the same plane p and whose Ith minor is the identity matrix. The nk entries in the columns not in the Ith minor of $(\tilde{p}_I)^{-1}\tilde{p}$ are local holomorphic coordinates near $p \in G_{n,n+k}(\mathbb{C})$.

Lemma 4. *Let $f : X \to G_{n,n+k}(\mathbb{C})$ be a continuous map. Then for any chart*

$$h : f^*(\gamma_n^*)|_U \to U \times \mathbb{C}^n \to \mathbb{C}^n$$

with $x \in U \subseteq X$, $f(x) \subseteq \mathbb{C}^{n+k}$ is spanned by the rows of the $n \times (n+k)$ matrix whose columns are $h(s_j^*(x)) \in \mathbb{C}^n$ where s_j^* is the pull-back of s_j along f for all $j = 1, \ldots, n+k$.

Proof:

Since $s_j^*(x) = (x, s_j(f(x)))$ for all $j = 1, \ldots, n+k$ we need only show that for any chart $\phi : \gamma_n^*|_U \to U \times \mathbb{C}^n \to \mathbb{C}^n$ with $f(x) \in U$ the $n \times (n+k)$ matrix whose columns are $\phi(s_j(f(x))) \in \mathbb{C}^n$ has rows which span $f(x) \in G_{n,n+k}(\mathbb{C})$.

A holomorphic chart around $f(x) \in G_{n,n+k}(\mathbb{C})$ is given by an $n \times (n+k)$ matrix of holomorphic functions whose rows $r_1(p), \ldots, r_n(p)$ span the n-plane $p \in G_{n,n+k}(\mathbb{C})$ for every point p in a neighborhood U of $f(x)$. These row vectors give a basis of the fiber of γ_n above every point $p \in U$ and hence induce a trivialization of $\gamma_n|_U$, i.e. if $(p, v) \in \gamma_n|_U$ satisfies

$$v = \sum_{j=1}^{n} a_j r_j(p)$$

for some $a_j \in \mathbb{C}$, then the trivialization $\gamma_n|_U \to U \times \mathbb{C}^n \to \mathbb{C}^n$ is defined by $(p, v) \mapsto (a_1, \ldots, a_n)$.

A framing of $\gamma_n^*|_U = \mathrm{Hom}(\gamma_n|_U, \mathbb{C})$ is given by the dual row vectors $r_1^*(p), \ldots, r_n^*(p)$ for all $p \in U$. As for $\gamma_n|_U$ this induces a trivialization of $\gamma_n^*|_U$, i.e. if $(p, v^*) \in \gamma_n^*|_U$ satisfies

$$v^* = \sum_{j=1}^{n} b_j r_j^*(p)$$

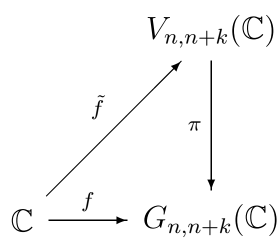
for some $b_j \in \mathbb{C}$, then the trivialization $\gamma_n^*|_U \to U \times \mathbb{C}^n \to \mathbb{C}^n$ is defined by $(p, v^*) \mapsto (b_1, \ldots, b_n)$. In this trivialization, the ith component of $s_j(f(x)) \in \mathrm{Hom}(\gamma_n|_{f(x)}, \mathbb{C})$ is

$$s_j(f(x))[(p, r_i(f(x))] = j\text{th coordinate of } r_i(f(x)) \in \mathbb{C}^{n+k}.$$

Since the lemma holds for this particular trivialization it also holds for any other compatible trivialization.

□

Theorem 5. *For every holomorphic map $f : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ there exists a polynomial map $\tilde{f} : \mathbb{C} \to V_{n,n+k}(\mathbb{C})$ such that the following diagram commutes:*



where $\mathbb{C}P^1 = \mathbb{C} \cup \infty$.

Proof:

The theorem follows immediately from the above lemma, Lemma 1, and the fact that every holomorphic n-plane bundle on $\mathbb{C}P^1$ splits holomorphically into a direct sum of line bundles.

□

3. The Bijection

Let $M_{n,n+k}(\mathbb{C}[z])$ be the set of $n \times (n+k)$ λ-matrices and let $GL_n(\mathbb{C}[z])$ be the set of $n \times n$ λ-matrices whose determinant is in $\mathbb{C}\backslash\{0\}$. $GL_n(\mathbb{C}[z])$ is a topological group that acts on $M_{n,n+k}(\mathbb{C}[z])$ by matrix multiplication on the left.

$$GL_n(\mathbb{C}[z]) \times M_{n,n+k}(\mathbb{C}[z]) \to M_{n,n+k}(\mathbb{C}[z])$$

Note that this action corresponds to polynomial row operations on an element of $M_{n,n+k}(\mathbb{C}[z])$. That is, by multiplying an element of $M_{n,n+k}(\mathbb{C}[z])$ on the left by an element of $GL_n(\mathbb{C}[z])$ we can interchange rows, multiply a row by a non-zero constant, or add a polynomial multiple of one row to another row. (For additional details see [2] Chapter 6.)

Let $P_{n,n+k}(\mathbb{C}[z])$ be the subset of $M_{n,n+k}(\mathbb{C}[z])$ consisting of those matrices whose rows are pointwise linearly independent. That is, those matrices whose rows are in the Stiefel manifold $V_{n,n+k}(\mathbb{C})$ when evaluated at every $z \in \mathbb{C}$. Another way of stating this condition is by requiring that the determinants of the $n \times n$ minors of a matrix in $P_{n,n+k}(\mathbb{C}[z])$ cannot all have a root in common. The space $P_{n,n+k}(\mathbb{C}[z])$ can be identified with the space of polynomial maps from $\mathbb{C}$ to $V_{n,n+k}(\mathbb{C})$. (Compare with Section 3.5 of [10].)

Claim 6. *The action of $GL_n(\mathbb{C}[z])$ on $M_{n,n+k}(\mathbb{C}[z])$ restricts to an action on $P_{n,n+k}(\mathbb{C}[z])$.*

$$GL_n(\mathbb{C}[z]) \times P_{n,n+k}(\mathbb{C}[z]) \to P_{n,n+k}(\mathbb{C}[z]).$$

Proof:

Let $M \in P_{n,n+k}(\mathbb{C}[z])$ and $G \in GL_n(\mathbb{C}[z])$. The determinants of the $n \times n$ minors of GM have the same roots as the determinants of the $n \times n$ minors of M since they differ only by a factor of $\det G \in \mathbb{C}$. This observation proves the claim since an $n \times (n+k)$ matrix of polynomials is in $P_{n,n+k}(\mathbb{C}[z])$ if and only if the determinants of its $n \times n$ minors do not all have a root in common.

□

Theorem 7. *The space of holomorphic maps $f : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ is in bijective correspondence with the orbit space of the action of $GL_n(\mathbb{C}[z])$ on $P_{n,n+k}(\mathbb{C}[z])$.*

$$Hol(\mathbb{C}P^1, G_{n,n+n}(\mathbb{C})) \longleftrightarrow P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$$

Proof:

In the previous section we showed that for every holomorphic map $f : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ there exists a λ-matrix, say $P \in P_{n,n+k}(\mathbb{C}[z])$, such that $f(z) = \pi(P(z))$ for all $z \in \mathbb{C}$ where $\pi : V_{n,n+k}(\mathbb{C}) \to G_{n,n+k}(\mathbb{C})$ is the map that sends an n-frame to the plane it spans. In order to show that this determines a well-defined element of the orbit space we must show that for any two elements $P_1, P_2 \in P_{n,n+k}(\mathbb{C}[z])$ that satisfy $\pi(P_1(z)) = \pi(P_2(z))$ for all $z \in \mathbb{C}$ there exists $G \in GL_n(\mathbb{C}[z])$ such that $GP_1 = P_2$.

Assume that $\pi(P_1(z)) = \pi(P_2(z))$ for all $z \in \mathbb{C}$. Then there exists a matrix of functions $G(z) = (g_{ij}(z))$ (i.e. $g_{ij} : \mathbb{C} \to \mathbb{C}$ for all $1 \le i, j \le n$) such that $G(z)P_1(z) = P_2(z)$ for all $z \in \mathbb{C}$. Since $P_1 \in P_{n,n+k}(\mathbb{C}[z])$ there exists a minor of P_1, say $(P_1)_I$, whose determinant is not the zero polynomial. For every $1 \leq j \leq n$ the jth row of G gives a system of n equations and n unknowns in $g_{1j}, g_{2j}, \ldots, g_{nj}$,

$$(g_{j1}, g_{j2}, \ldots, g_{jn})(P_1)_I = (p_{j1}, p_{j2}, \ldots, p_{jn})$$

where $p_{j1}, p_{j2}, \ldots, p_{jn}$ are the entries in the jth row of the minor $(P_2)_I$. The above is a linear system of n equations and n unknowns over the field of rational functions. Moreover, since the determinant of $(P_1)_I$ is not zero this system of equations has a solution over the field of rational functions. That is, the functions g_{ij} are rational functions which are defined for all $z \in \mathbb{C}$, i.e. the g_{ij} are polynomials. This shows that $G \in GL_n(\mathbb{C}[z])$ since $\det G(z) \neq 0$ for all $z \in \mathbb{C}$. Therefore we have a well defined map

$$\mathrm{Hol}(\mathbb{C}P^1, G_{n,n+n}(\mathbb{C})) \longrightarrow P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z]).$$

To show that this map has an inverse we need only show that for every orbit there exists a λ-matrix P in the orbit such that the map defined by $f(z) = \pi(P(z))$ is holomorphic for $z \in \mathbb{C}$ and extends continuously to the point at infinity. Then $\infty = [0 : 1] \in \mathbb{C}P^1$ will be a removable singularity and we will have a holomorphic map $f : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ defined which clearly corresponds to the orbit of P. If we embed $G_{n,n+k}(\mathbb{C})$ into $\mathbb{C}P^N$ using the Plücker embedding, then the map $f : \mathbb{C} \to G_{n,n+k}(\mathbb{C}) \hookrightarrow \mathbb{C}P^N$ is given by $N+1$ polynomials and it's clear that a continuous extension to ∞ is simply given by the coefficients of the highest power of z in these $N+1$ polynomials. Since $G_{n,n+k}(\mathbb{C}) \hookrightarrow \mathbb{C}P^N$ is a closed subset this point must be contained in $G_{n,n+k}(\mathbb{C})$, and hence we have defined a holomorphic map $f : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ that satisfies $f(z) = \pi(P(z))$ for all $z \in \mathbb{C}$.

□

4. Topological Issues

The space $\mathrm{Hol}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is given the compact-open topology. Since $G_{n,n+k}(\mathbb{C})$ is a metric space, the compact-open topology on $\mathrm{Hol}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is the same as the topology of compact convergence (see [9] p. 286). Moreover, since $\mathbb{C}P^1$ is compact a sequence of holomorphic maps $f_j \in \mathrm{Hol}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ converges to $f \in \mathrm{Hol}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ if and only if for every $\epsilon > 0$ there exists $J \in \mathbb{N}$ such that

$$\sup\{d(f_j(z), f(z)) | z \in \mathbb{C}P^1\} < \epsilon$$

for all $j > J$ where d denotes the metric on $G_{n,n+k}(\mathbb{C})$. (For more details see [9] p. 280-283.)

The space $P_{n,n+k}(\mathbb{C}[z])$ is topologized as a subspace of the vector space $\mathbb{C}[z]^{n(n+k)}$, and the orbit space $P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ is given the quotient topology. The following lemma gives a good intuitive way to understand the topology of $P_{n,n+k}(\mathbb{C})/GL_n(\mathbb{C}[z])$.

Lemma 8. *Let G be a topological group and assume that G acts continuously on a topological space X*

$$G \times X \to X$$

with quotient map $\pi : X \to X/G$. Then π is an open map and a sequence of equivalence classes $\bar{x}_j \in X/G$ converges to $\bar{x} \in X/G$ as $j \to \infty$ if and only if there exists a sequence $x_j \in X$ and an $x \in X$ such that $\pi(x_j) = \bar{x}_j$ for all $j \in \mathbb{N}$, $\pi(x) = \bar{x}$, and $x_j \to x$ as $j \to \infty$.

Proof:

For proof that π is an open map see [6] p. 36. The essential point is that for any open set $U \subseteq X$

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U.$$

Now assume that $x_j \to x \in X$ as $j \to \infty$. Since π is continuous we have $\pi(x_j) \to \pi(x)$ as $j \to \infty$. For the other direction assume that we have a sequence $\bar{x}_j \in X/G$, a point $\bar{x} \in X/G$, and an open set U_α containing x_α for each $x_\alpha \in \pi^{-1}(\bar{x})$ such that $\pi^{-1}(\bar{x}_j) \cap U_\alpha = \emptyset$ for all α, j. Then $\pi(\cup_\alpha U_\alpha)$ is an open set containing $\bar{x}$ but not $\bar{x}_j$ for all $j \in \mathbb{N}$. Therefore $\bar{x}_j$ does not converge to $\bar{x}$.

□

To show that the bijection defined in the previous section is a homeomorphism when restricted to maps of a fixed degree, we will reduce the problem to one of maps between projective spaces using the Plücker embedding. Let $N = \binom{n+k}{n}$. The Plücker embedding $Pl : G_{n,n+k}(\mathbb{C}) \to \mathbb{C}P^{N-1}$ is defined by sending a plane to the homogeneous coordinates given by the determinants of the $n \times n$ minors of any element of $V_{n,n+k}(\mathbb{C})$ whose rows span the plane. We have a similar map

$$Pl : P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z]) \to \mathbb{P}(\mathbb{C}[z]^N)$$

defined by sending an equivalence class $[M]$ to the N-tuple of polynomials (mod $\mathbb{C}^*$) given by the determinants of the $n \times n$ minors of M. This generalized Plücker embedding is well-defined because multiplying M by an element of $GL_n(\mathbb{C}[z])$ can only change the determinants of the $n \times n$ minors of M by an element of $\mathbb{C}^*$.

Lemma 9. $Pl : P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z]) \to \mathbb{P}(\mathbb{C}[z]^N)$ *is an embedding.*

Proof:

Assume that the determinants of the $n \times n$ minors of $M_1, M_2 \in P_{n,n+k}(\mathbb{C}[z])$ are the same up to multiplication by an element of $\mathbb{C}^*$. Since the standard Plücker embedding $G_{n,n+k}(\mathbb{C}) \to \mathbb{C}P^{N-1}$ is injective, there exists a matrix of functions $G(z) = (g_{ij}(z))$ (i.e. $g_{ij} : \mathbb{C} \to \mathbb{C}$ for all $1 \le i, j \le n$) such that $G(w)M_1(w) = M_2(w)$ for all $w \in \mathbb{C}$. Since $M_1 \in P_{n,n+k}(\mathbb{C}[z])$ there exists a minor of M_1, say $(M_1)_I$, whose determinant is not the zero polynomial. For every $1 \le j \le n$ the jth row of G gives a system of n equations and n unknowns in $g_{1j}, g_{2j}, \ldots, g_{nj}$,

$$(g_{j1}, g_{j2}, \ldots, g_{jn})(M_1)_I = (l_{j1}, l_{j2}, \ldots, l_{jn})$$

where $l_{j1}, l_{j2}, \ldots, l_{jn}$ are the entries in the jth row of the minor $(M_2)_I$. The above system of equations is a linear system of n equations and n unknowns over the field of rational functions. Moreover, since the determinant of $(M_1)_I$ is not zero this system of equations has a solution over the field of rational functions. Hence the functions g_{ij} are rational functions that have no poles, i.e. polynomials. Therefore, $G \in GL_n(\mathbb{C}[z])$ and Pl is injective.

The following commutative diagram shows that Pl is continuous.

$$\begin{array}{ccc} P_{n,n+k}(\mathbb{C}[z]) & \xrightarrow{det \times \cdots \times det} & \mathbb{C}[z]^N \\ \downarrow & & \downarrow \\ P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z]) & \xrightarrow{Pl} & \mathbb{P}(\mathbb{C}[z]^N) \end{array}$$

To see that the inverse map is continuous it suffices to show that the composite

$$\begin{array}{ccc} P_{n,n+k}(\mathbb{C}[z]) & & \\ \downarrow{\scriptstyle \pi} & & \\ P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z]) & \xrightarrow{Pl} & \mathbb{P}(\mathbb{C}[z]^N) \end{array}$$

maps open sets to open sets in its image.

Every point $M \in P_{n,n+k}(\mathbb{C}[z])$ has an open neighborhood given by perturbing the coefficients of the entries of M by $\pm\epsilon$ which maps onto an open neighborhood of $(Pl \circ \pi)(M)$. That is, $(Pl \circ \pi)(M) \in \mathbb{P}(\mathbb{C}[z]^N)$ has homogeneous coordinates which are linear functions in the coefficients of the polynomial entries of M. Since a linear function of several variables is an open map $Pl \circ \pi$ is an open map.

$\square$

Theorem 10. *The map*

$$\phi : Hol(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \longrightarrow P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$$

which sends a holomorphic map to the equivalence class of the λ-matrix P such that $f(z) = \pi(P(z))$ for all $z \in \mathbb{C}$ is continuous. When ϕ is restricted to maps of a fixed degree it is a homeomorphism onto its image.

Proof:

Let $N = \binom{n+k}{n}$. The following diagram commutes.

$$\begin{array}{ccc} \mathrm{Hol}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) & \xrightarrow{\phi} & P_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z]) \\ \Big\downarrow{\scriptstyle \circ Pl} & & \Big\downarrow{\scriptstyle Pl} \\ \mathrm{Hol}(\mathbb{C}P^1, \mathbb{C}P^N) & \xrightarrow{\phi} & \mathbb{P}(\mathbb{C}[z]^N) \end{array}$$

Hence by the preceeding lemma it suffices to prove the theorem for the case $n = 1$ since the restriction of a continuous map is continuous.

$\mathrm{Hol}(\mathbb{C}P^1, \mathbb{C}P^N)$ has countably many components. The components are distinguished by the topological degrees of the maps. So it suffices to show that a sequence $f_j \in \mathrm{Hol}(\mathbb{C}P^1, \mathbb{C}P^N)$ of fixed degree d converges to $f \in \mathrm{Hol}(\mathbb{C}P^1, \mathbb{C}P^N)$ if and only if $\phi(f_j) \to \phi(f)$ as $j \to \infty$. Suppose that in homogeneous coordinates

$$f_j(z) = (p_j^0(z) : p_j^1(z) : \cdots : p_j^N(z))$$

and

$$f(z) = (p^0(z) : p^1(z) : \cdots : p^N(z)).$$

Since f_j and f are of degree d for all j we may assume that $p_j^0(z)$ and $p^0(z)$ are monic polynomials of degree d for all j. This means that $\phi(f_j)$ and $\phi(f)$ all lie in a single coordinate chart of $\mathbb{P}(\mathbb{C}[z]^N)$. Hence $\phi(f_j)$ converges to $\phi(f)$ if and only if for all $0 \leq i \leq N$ the coefficients of $p_j^i(z)$ converge to the coefficients of $p^i(z)$ as $j \to \infty$.

Assume that $f_j(z)$ converges to $f(z)$ uniformly for all $z \in \mathbb{C}P^1$. This implies that for every $0 \leq i \leq N$ and for a generic $z \in \mathbb{C}$ (where $p_j^0(z) \neq 0$ and $p^0(z) \neq 0$)

$$\lim_{j \to \infty} \frac{p_j^i(z)}{p_j^0(z)} = \frac{p^i(z)}{p^0(z)}$$

Therefore the coefficients of $p_j^i(z)$ converge to the coefficients of $p^i(z)$ for all $0 \leq i \leq N$.

Now assume that for all $0 \leq i \leq N$ the coefficients of $p_j^i(z)$ converge to the coefficients of $p^i(z)$. We want to show that for every $\epsilon > 0$ there exists a J such that for all $j > J$

$$\sup\{d(f_j(z), f(z)) | z \in \mathbb{C}P^1\} < \epsilon$$

where $d(-,-)$ denotes the metric on $\mathbb{C}P^N$. There are several ways of describing this metric. One way is to take the angle between two lines in $\mathbb{C}^{N+1}$ as the metric. An equivalent choice is to take the Hermitian inner product of a unit vector in the first line with a unit vector in the orthogonal complement of the second line. For instance:

$$\frac{\sum_{i=0}^{(N-1)/2} p_j^{2i}(z)\overline{p^{2i+1}(z)} - \overline{p_j^{2i+1}(z)}p^{2i}(z)}{||(p_j^0(z) : p_j^1(z) : \cdots : p_j^N(z))|| \; ||(p^0(z) : p^1(z) : \cdots : p^N(z))||}$$

(Here we have assumed that N is odd. If N is even, then embed $\mathbb{C}P^N$ in $\mathbb{C}P^{N+1}$ by taking the last coordinate to be zero.)

Pick any $\epsilon > 0$. For any closed disk $D(r)$ of radius r the above expression shows that for all $z \in D(r)$ there exists a J_1 such that for all $j > J_1$ we have $d(f_j(z), f(z)) < \epsilon/2$. If r is large then the polynomials will behave like their highest terms when $|z| > r$, and hence it is possible to pick a J_2 such that for all $j > J_2$ we have $d(f_j(z), f(z)) < \epsilon/2$ for all $|z| > r$. Taking $J = \max\{J_1, J_2\}$ we see that $f_j(z)$ converges to $f(z)$ uniformly for all $z \in \mathbb{C}P^1$.

□

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Department of Mathematics, Penn State University – The Altoona College, 101B Eiche, Altoona, PA 16601-3760

E-mail address: hurtubis@math.psu.edu