THE FLOW CATEGORY OF THE ACTION FUNCTIONAL ON $\mathcal{L}G_{n,n+k}(\mathbb{C})$

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Abstract. The flow category of a Morse-Bott-Smale function $f_A : G_n(\mathbb{C}^\infty) \to \mathbb{R}$ is shown to be related to the flow category of the action functional on the universal cover of $\mathcal{L}G_{n,n+k}(\mathbb{C})$ via a group action. The Floer homotopy type and the associated cohomology ring of $f_A : G_n(\mathbb{C}^\infty) \to \mathbb{R}$ are computed. When $n = 1$ this cohomology ring is the Floer cohomology of $G_{1,1+k}(\mathbb{C})$.

1. Introduction

In [8] Floer defined cohomology groups associated to a perturbed action functional on the loop space of a monotone symplectic manifold. In related papers Floer defined cohomology groups for the Chern Simons' functional on a 3-manifold [6] and for the intersection of Lagrangian submanifolds [7]. Floer's cohomology groups were defined using infinite dimensional Morse theoretic techniques, but in several aspects his methods were fundamentally different from those in traditional infinite dimensional Morse theory. For example, the critical points in his theory all have infinite index (although the relative index between any two critical points is finite); also, the higher dimensional spaces of piecewise gradient flow lines in Floer's theory may be non-compact.

Recently Cohen, Jones, and Segal have been studying the properties that a function on an infinite dimensional manifold must have in order to define Floer cohomology. Furthermore, in several cases they have studied, the "Floer function," i.e. a function which can be used to define Floer cohomology groups, can actually be used to define an inverse system of spectra (a pro-spectrum). They call this inverse system of spectra the "Floer homotopy type" and the Floer cohomology groups can be recovered from the Floer homotopy type. One of their goals is to discover what additional properties (if any) a Floer function must satisfy in order to define the Floer homotopy type.

A basic component of their theory is that one can encode the dynamics of a Floer function in terms of a topological category which

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they call the "flow category." The objects of the flow category are the critical points of the function and the morphisms are the unparameterized piecewise gradient flow lines of the function. In finite dimensions the flow category is compact (i.e. the morphism spaces are compact) and framed (i.e. there is a stable framing of each morphism space). The geometric realization of the flow category of an arbitrary Morse function on a finite dimensional compact manifold is homotopy equivalent to the manifold, and if the function satisfies the Morse-Smale transversality condition then the geometric realization of the flow category is homeomorphic to the manifold [4]. In infinite dimensions the Floer homotopy type is constructed from the flow category of the action functional on a covering of the manifold. For instance, the Floer homotopy type of the action functional on $\mathcal{L}\mathbb{C}P^k$ is constructed from the flow category of the action functional on the universal cover of $\mathcal{L}\mathbb{C}P^k$.

In [5] Cohen, Jones, and Segal announced some results concerning the Floer homotopy type of the action functional on $\mathcal{L}\mathbb{C}P^k$. In this paper I generalize their results by proving a theorem that relates the flow category of the action functional on the universal cover of $\mathcal{L}G_{n,n+k}(\mathbb{C})$ to the flow category of a Morse-Bott function on $G_n(\mathbb{C}^\infty)$. One difficulty which arises with the flow category of the action functional on the universal cover of $\mathcal{L}G_{n,n+k}(\mathbb{C})$ is that the morphism spaces are not compact. In [5] Cohen, Jones, and Segal note that the Donaldson-Uhlenbeck compactification of the space of (parameterized) gradient flow lines of the action functional between any two critical submanifolds in the universal cover of $\mathcal{L}\mathbb{C}P^k$ (i.e. the space of holomorphic maps $\mathbb{C}P^1 \rightarrow \mathbb{C}P^k$ of degree d) is homeomorphic to $\mathbb{C}P^{(k+1)(d+1)-1}$. Moreover, they note that for every $k \in \mathbb{Z}_+$ one can construct a natural compactification of the flow category by embedding it into the flow category of a Morse-Bott-Smale function $f_A : \mathbb{C}P^\infty \rightarrow \mathbb{R}$. The Floer homotopy type of the compactified flow category of the action functional on $\mathcal{L}\mathbb{C}P^k$ is

$$\mathbb{C}P^\infty \leftarrow (\mathbb{C}P^\infty)^{-(1+k)\gamma_1} \leftarrow (\mathbb{C}P^\infty)^{-2(1+k)\gamma_1} \leftarrow \cdots$$

where γ_1 denotes the Hopf line bundle over $\mathbb{C}P^\infty$.

When $n > 1$ there is a Morse-Bott-Smale function $f_A : G_n(\mathbb{C}^\infty) \rightarrow \mathbb{R}$ (for every $k \in \mathbb{Z}_+$) which generalizes the function $f_A : \mathbb{C}P^\infty \rightarrow \mathbb{R}$ used by Cohen, Jones, and Segal, but the flow category of the action functional on the universal cover of $\mathcal{L}G_{n,n+k}(\mathbb{C})$ does not embed into the flow category of this function. There is however an $\mathbb{R}$-equivariant fiber bundle which relates the two flow categories. In this paper I prove that there exists a Morse-Bott-Smale function $f_A : G_n(\mathbb{C}^\infty) \rightarrow \mathbb{R}$

such that for an open dense subset $\mathcal{U} \subseteq V_n(\mathbb{C}^\infty)$ the topological group $GL_n^c(\mathbb{C}[z, z^{-1}])$, consisting of $n \times n$ matrices with Laurent polynomial entries whose determinant is a non-zero constant, acts on $\mathcal{U}$ and induces a flow category on $\mathcal{U}/GL_n^c(\mathbb{C}[z, z^{-1}])$ from the gradient flow lines of $f_A : G_n(\mathbb{C}^\infty) \to \mathbb{R}$. The induced flow category is isomorphic to the flow category of the action functional on the universal cover of $\mathcal{L}G_{n,n+k}(\mathbb{C})$. That is we have the following fiber bundle,

$$\begin{array}{ccc} GL_n^c(\mathbb{C}[z, z^{-1}])/GL_n(\mathbb{C}) & \longrightarrow & \mathcal{U}/GL_n(\mathbb{C}) \\ & & \big\downarrow \pi \\ & & \mathcal{U}/GL_n^c(\mathbb{C}[z, z^{-1}]) \end{array}$$

whose projection map π is $\mathbb{R}$-equivariant with respect to the restriction of the gradient flow of $f_A : G_n(\mathbb{C}^\infty) \to \mathbb{R}$ to $\mathcal{U}/GL_n(\mathbb{C}) \subseteq G_n(\mathbb{C}^\infty)$. Later in this paper $\mathcal{U} \subseteq V_n(\mathbb{C}^\infty)$ is identified as the space of all polynomial maps $\mathbb{C}^* \to V_{n,n+k}(\mathbb{C})$. Note that when $n = 1$ we have $\pi = id$ and this reduces to the result announced by Cohen, Jones and Segal in [5].

The Floer homotopy type of $f_A : G_n(\mathbb{C}^\infty) \to \mathbb{R}$ has

$$G_n(\mathbb{C}^\infty) \leftarrow G_n(\mathbb{C}^\infty)^{-(n+k)\gamma_n} \leftarrow G_n(\mathbb{C}^\infty)^{-2(n+k)\gamma_n} \leftarrow \cdots$$

as a cofinal system where γ_n denotes the tautological n-plane bundle over $G_n(\mathbb{C}^\infty)$ and the maps are induced by certain bundle inclusions. Applying H^* and using the Thom Isomorphism Theorem we have the following direct system.

$$H^*(G_n(\mathbb{C}^\infty)) \stackrel{\cup c_n^{n+k}}{\longrightarrow} H^{*+2n(n+k)}(G_n(\mathbb{C}^\infty)) \stackrel{\cup c_n^{n+k}}{\longrightarrow} \cdots$$

The direct limit of this system is $H^*(G_n(\mathbb{C}^\infty))[c_n^{-(n+k)}]$. When $n = 1$ this ring is the Floer cohomology of $\mathbb{C}P^k$.

2. Flow Categories

The flow category of a Morse function on a finite dimensional compact smooth Riemannian manifold M was first defined by Cohen, Jones, and Segal in [3]. As they note in [5] their definition readily extends to a Morse-Bott function [2] on a finite dimensional compact smooth Riemannian manifold.

Definition 1. *Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a finite dimensional compact smooth Riemannian manifold M. The flow category of f, denoted $\mathcal{C}_f$, is the topological category whose objects are the* *critical points of f topologized as a subspace of M and whose morphisms are the unparameterized piecewise gradient flow lines of f. That is, for any two critical points a and b, $Mor(a,b)$ is defined to be the space of all continuous maps $\omega : [f(b), f(a)] \to M$ satisfying*

1. $\omega(f(b)) = b$
2. $\omega(f(a)) = a$
3. *Away from the critical points of f_A the map ω is smooth and satisfies the following differential equation.*

$$\frac{d\omega}{dt} = \frac{\nabla(f)}{\|\nabla(f)\|^2}$$

$Mor(a,b)$ is topologized as a subset of the space of all continuous maps from the closed interval $[f(b), f(a)]$ to M. This space of continuous maps is given the compact-open topology. Composition in $\mathcal{C}_f$ is given by concatenation.

In [3] Cohen, Jones, and Segal prove the following theorem for a Morse function f defined on a finite dimensional compact smooth Riemannian manifold M, and in [5] they note that their proof generalizes to the case when f is Morse-Bott.

Definition 2. *Let $f : M \to \mathbb{R}$ be a Morse-Bott function defined on a smooth Riemannian manifold M. f is said to satisfy the Morse-Bott-Smale transversality condition if and only if for any two critical submanifolds M and N, $W^u(m) \pitchfork W^s(N)$ for all $m \in M$.*

Let $B\mathcal{C}_f$ denote the geometric realization of $\mathcal{C}_f$.

Theorem 3.
(1) If $f : M \to \mathbb{R}$ is a generic Morse-Bott function (one whose gradient flow satisfies the Morse-Bott-Smale transversality condition) then there is a homeomorphism

$$B\mathcal{C}_f \cong M.$$

(2) For any Morse-Bott function $f : M \to \mathbb{R}$ there is a homotopy equivalence

$$B\mathcal{C}_f \simeq M.$$

The above definition of the flow category is sufficient for finite dimensional compact manifolds, but in infinite dimensions the equation

$$\frac{d\omega}{dt} = \frac{\nabla(f)}{\|\nabla(f)\|^2}$$

may not give a well posed initial value problem. Moreover, we would prefer a definition of the flow category which makes sense in the more general setting of an $\mathbb{R}$-action on a space X where X is not necessarily a manifold. The following definition is general enough to apply to a wide variety of problems.

Definition 4. *Let X be a metric space with an action $\mathbb{R} \times X \to X$. Let $a, b \in Ob(\mathcal{C}_X)$. Define $Mor'(a,b)$ to be the space of all piecewise flow lines on X from a to b, i.e. $\mathbb{R}$-equivariant subsets of X which are the images of continuous injective paths from a to b. The topology on $Mor'(a,b)$ is the topology induced from the $\sup - \inf$-metric d_{si}, i.e. if $l_1, l_2 \in Mor'(a,b)$, then*

$$d_{si}(l_1, l_2) = \sup_{x_1 \in l_1} \inf_{x_2 \in l_2} d(x_1, x_2) + \sup_{x_2 \in l_2} \inf_{x_1 \in l_1} d(x_1, x_2)$$

where d is the metric on X.

Theorem 5. *For a finite dimensional compact smooth Riemannian manifold M there is a homeomorphism*

$$\phi : Mor(a,b) \to Mor'(a,b)$$

for any $a, b \in Ob(\mathcal{C}_f)$ defined by sending a map in $Mor(a,b)$ to its image.

Proof: It is clear that ϕ is a bijection. Let $\omega_1, \omega_2 \in Mor(a,b)$. Since

$$d_{si}(\phi(\omega_1), \phi(\omega_2)) \leq d_{\sup}(\omega_1, \omega_2)$$

ϕ is continuous.

Now assume that $l_j \to l \in Mor'(a,b)$. To show that ϕ^{-1} is continuous it suffices to show that $\phi^{-1}(l_j) \to \phi^{-1}(l)$. To prove this we will use the fact that for a compact finite dimensional manifold M the space $Mor(a,b)$ is compact [3]. Ascoli's Theorem then implies that $Mor(a,b)$ is uniformly equicontinuous.

Pick any $\epsilon > 0$. Choose $\delta > 0$ such that

$$|t_1 - t_2| < \delta \text{ implies } d(\omega(t_1), \omega(t_2)) < \epsilon/2$$

for all $t_1, t_2 \in [f(b), f(a)]$ and for all $\omega \in Mor(a,b)$. Since f is uniformly continuous there exists $\delta_1 > 0$ such that

$$d(x_1, x_2) < \delta_1 \text{ implies } |f(x_1) - f(x_2)| < \delta$$

for all $x_1, x_2 \in M$. Choose J such that $j > J$ implies

$$d_{si}(l_j, l) < \min\{\delta_1, \epsilon/2\}.$$

Then for any $t \in [f(b), f(a)]$ and for all $j > J$ we have

$$d(\phi^{-1}(l_j)(t), \phi^{-1}(l)(t)) < d(\phi^{-1}(l)(t), x_j) + d(x_j, \phi^{-1}(l_j)(t))$$

where $x_j \in l_j$ is the point closest to $\phi^{-1}(l)(t)$. Since $d(\phi^{-1}(l)(t), x_j) < \min\{\delta_1, \epsilon/2\}$ and $f(\phi^{-1}(l)(t)) = t$ we have

$$|t - f(x_j)| < \delta$$

which implies $d(x_j, \phi^{-1}(l_j)(t)) < \epsilon/2$. Therefore,

$$d(\phi^{-1}(l_j)(t), \phi^{-1}(l)(t)) < \epsilon/2 + \epsilon/2$$

for all $t \in [f(b), f(a)]$ and for all $j > J$.

$\square$

The assumption that X is metrizable is not essential. Given any topological space X with an action $\mathbb{R} \times X \to X$ we can take as a basis for the topology of $Mor'(a, b)$ the sets

$$B(U_1, \ldots, U_n) = \{l \in Mor'(a, b) | l \cap U_j \neq \emptyset \text{ for all } j = 1, \ldots, n\}$$

where $U_1, \ldots, U_n$ are open sets in X. It is easy to see that the topology defined by this basis agrees with the topology defined above on $Mor'(a, b)$ when X is a metric space.

From now on we will define the flow category $\mathcal{C}_f$ by taking $Mor'(a, b)$ as the space of morphisms from a to b. The main advantage to this approach is that the flow category is now defined for every topological space X with an action $\mathbb{R} \times X \to X$. The following theorem follows immediately from the definition.

Theorem 6. *Let X and Y be $\mathbb{R}$-spaces and let $g : X \to Y$ be an $\mathbb{R}$-equivariant map. Then g induces a functor $G : \mathcal{C}_X \to \mathcal{C}_Y$. If g is continuous, then G is continuous.*

3. The relationship between the two flow categories

Let $(G_{n,n+k}(\mathbb{C}), \omega)$ denote the complex Grassmann manifold of n-planes in $\mathbb{C}^{n+k}$ with its standard symplectic form ω. Since $G_{n,n+k}(\mathbb{C})$ is simply connected, $\pi_j(\mathcal{L}G_{n,n+k}(\mathbb{C})) = \pi_{j+1}(G_{n,n+k}(\mathbb{C}))$ for all $j \in \mathbb{Z}_+$ where $\mathcal{L}G_{n,n+k}(\mathbb{C})$ denotes the free loop space. The universal cover of the free loop space consists of equivalence classes $[\gamma, \omega]$ where $\gamma : S^1 \to G_{n,n+k}(\mathbb{C})$ is in $\mathcal{L}G_{n,n+k}(\mathbb{C})$ and $\theta : D^2 \to G_{n,n+k}(\mathbb{C})$ is an extension of γ well defined up to homotopy rel S^1. The action functional $\tilde{\mathcal{A}}_\omega : \widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C}) \to \mathbb{R}$ is defined by

$$\tilde{\mathcal{A}}_\omega([\gamma]) = \int_{D^2} \theta^* \omega.$$

This descends to a function $\mathcal{A}_\omega : \mathcal{L}G_{n,n+k}(\mathbb{C}) \to \mathbb{R}/\mathbb{Z}$.

In [8] Floer defined cohomology groups graded mod $2N$ for a monotone symplectic manifold (M, ω) where N is the minimal Chern number of M. Floer's chain complex is generated by the critical points of a perturbation of the action functional on M. He defined an index for these critical points which is well defined mod $2N$. In [17] Salamon and Zehnder defined a Maslov index for the critical points of a perturbation of the action functional on the universal abelian cover of a symplectic manifold (M,ω). Their index can be used to define Floer cohomology groups graded over $\mathbb{Z}$ and periodic with period $2N$ (see section 10.1 of [14]). The universal abelian cover of M is the covering space of $\mathcal{L}M$ whose group of deck transformations is the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$ modulo torsion. Since $G_{n,n+k}(\mathbb{C})$ is simply connected and $\pi_2(G_{n,n+k}(\mathbb{C})) = \mathbb{Z}$ is torsion free, the universal abelian cover of $G_{n,n+k}(\mathbb{C})$ is the universal cover $\widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})$.

The boundary operator in Floer's chain complex is defined by counting the number of gradient flow lines of the perturbed action functional on $\mathcal{L}M$ connecting two critical points. In [14] McDuff and Salamon define the boundary operator by counting the number of gradient flow lines of the perturbed action functional on the universal abelian cover of $\mathcal{L}M$ connecting two critical points. The projection from the universal abelian cover of $\mathcal{L}M$ to $\mathcal{L}M$ is $\mathbb{R}$-equivariant and hence maps critical points to critical points and gradient flow lines to gradient flow lines.

In this section we will study the flow category of the unperturbed action functional on the universal cover of $\mathcal{L}G_{n,n+k}(\mathbb{C})$. The gradient flow lines are lifts of the gradient flow lines of the action functional $\mathcal{A}_\omega : \mathcal{L}G_{n,n+k}(\mathbb{C}) \to \mathbb{R}/\mathbb{Z}$. Floer cohomology for the unperturbed action functional on the universal cover of a symplectic manifold was defined using spectral sequences by Ruan and Tian in [16].

Definition 7. *Define $\gamma \in \mathcal{L}G_{n,n+k}(\mathbb{C})$ (or $\widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})$) to be an algebraic point if and only if γ lies on a gradient flow line which begins and ends at critical points.*

The gradient flow lines of $\mathcal{A}_\omega$ are holomorphic curves from $\mathbb{R} \times S^1 = \mathbb{C}/i\mathbb{Z}$ to $G_{n,n+k}(\mathbb{C})$ [8]. Thus $\gamma \in \mathcal{L}G_{n,n+k}(\mathbb{C})$ is an algebraic point if and only if there exists a holomorphic curve $h : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ such that $h(e^{is}) = \gamma(s)$ for all $s \in [0, 2\pi]$. A holomorphic curve $h : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ determines a gradient flow line $h' : \mathbb{R} \to \mathcal{L}G_{n,n+k}(\mathbb{C})$ by $h'(t)(s) = h(e^{t+is})$ and these flow lines lift to gradient flow lines on $\widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})$. It is the images of these lifts which determine the flow category of $\tilde{\mathcal{A}}_\omega : \widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C}) \to \mathbb{R}$.

Definition 8. *The flow category of $\mathcal{A}_\omega : \mathcal{L}G_{n,n+k}(\mathbb{C}) \to \mathbb{R}/\mathbb{Z}$ (or $\tilde{\mathcal{A}}_\omega$) is defined to be the flow category of the space of algebraic points in $\mathcal{L}G_{n,n+k}(\mathbb{C})$ (or $\widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})$) where the $\mathbb{R}$-action is given by the gradient flow.*

The object space of the flow category of $\mathcal{A} : \mathcal{L}G_{n,n+k}(\mathbb{C}) \to \mathbb{R}$ consists of a single critical submanifold, $G_{n,n+k}(\mathbb{C}) \subseteq \mathcal{L}G_{n,n+k}(\mathbb{C})$. Since

$G_{n,n+k}(\mathbb{C})$ is simply connected and $\pi_1(\mathcal{L}G_{n,n+k}(\mathbb{C})) = \mathbb{Z}$, the object space of the flow category of $\tilde{\mathcal{A}} : \widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C}) \to \mathbb{R}$ is $\mathbb{Z} \times G_{n,n+k}(\mathbb{C})$. A gradient flow line of the action functional $\mathbb{R} \to \mathcal{L}G_{n,n+k}(\mathbb{C})$ which begins and ends at critical points is given by a holomorphic map $h : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$. Such maps can be represented by equivalence classes of $n \times (n+k)$ matrices with polynomial entries (see for instance [13]). As we will see later in this section, the lifts of the corresponding gradient flow lines to $\widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})$ can be represented by equivalence classes of $n \times (n + k)$ matrices with Laurent polynomial entries.

Let $\mathbb{C}[z, z^{-1}]$ be the ring of Laurent polynomials. As a vector space over $\mathbb{C}$ we have

$$\mathbb{C}[z, z^{-1}] \approx \mathbb{C}^\infty.$$

We will use the notation $V_{n,n+k}(\mathbb{C}[z, z^{-1}])$ to denote the Stiefel manifold of n-tuples of linearly independent vectors in the infinite dimensional complex vector space

$$\mathbb{C}[z, z^{-1}]^{n+k} = \underbrace{\mathbb{C}[z, z^{-1}] \times \cdots \times \mathbb{C}[z, z^{-1}]}_{n+k} \approx \mathbb{C}^\infty$$

and

$$G_{n,n+k}(\mathbb{C}[z, z^{-1}]) = V_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n(\mathbb{C}) \approx G_n(\mathbb{C}^\infty)$$

to denote the infinite dimensional complex Grassmann manifold of n-planes in $\mathbb{C}[z, z^{-1}]^{n+k}$.

Let $M_{n,n+k}(\mathbb{C})$ be the set of all $n \times (n + k)$ matrices with entries in $\mathbb{C}$ and let $M_{n,n+k}(\mathbb{C}[z, z^{-1}])$ be the set of all $n \times (n + k)$ matrices with entries in $\mathbb{C}[z, z^{-1}]$. For every $w \in \mathbb{C}^*$ we have an evaluation map $e_w : V_{n,n+k}(\mathbb{C}[z, z^{-1}]) \to M_{n,n+k}(\mathbb{C})$ defined by evaluating the Laurent polynomial entries of $M \in V_{n,n+k}(\mathbb{C}[z, z^{-1}])$ at the point w. We define

$$\mathcal{U} = P_{n,n+k}(\mathbb{C}[z, z^{-1}]) = \bigcap_{w \in \mathbb{C}^*} e_w^{-1}(V_{n,n+k}(\mathbb{C}))$$

to be the elements of $V_{n,n+k}(\mathbb{C}[z, z^{-1}])$ which are pointwise linearly independent on $\mathbb{C}^*$. In other words, $P_{n,n+k}(\mathbb{C}[z, z^{-1}])$ is the space of polynomial maps $\mathbb{C}^* \to V_{n,n+k}(\mathbb{C})$ (see Section 3.5 of [15]).

Let $GL_n(\mathbb{C}[z, z^{-1}])$ be the group of all $n \times n$ matrices with Laurent polynomial entries whose determinant is invertible in $\mathbb{C}[z, z^{-1}]$ and let $GL_n^c(\mathbb{C}[z, z^{-1}])$ be the subgroup consisting of those matrices whose determinant is a non-zero constant. The main theorem in this paper can now be stated precisely as follows.

Theorem 9. *The flow category of* $\tilde{\mathcal{A}} : \widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C}) \to \mathbb{R}$ *is isomorphic to a flow category on* $P_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n^c(\mathbb{C}[z, z^{-1}])$ *induced from* *the gradient flow of a Morse-Bott-Smale function* $f_A : G_n(\mathbb{C}^\infty) \to \mathbb{R}$. $P_{n,n+k}(\mathbb{C}[z, z^{-1}])$ *is an open dense subset of* $V_{n,n+k}(\mathbb{C}[z, z^{-1}])$ *and the flow category on the orbit space* $P_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n^c(\mathbb{C}[z, z^{-1}])$ *is induced via the following* $\mathbb{R}$*-equivariant fiber bundle.*

$$\begin{array}{ccc} GL_n^c(\mathbb{C}[z, z^{-1}])/GL_n(\mathbb{C}) & \longrightarrow & P_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n(\mathbb{C}) \\ & & \Big\downarrow \pi \\ & & P_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n^c(\mathbb{C}[z, z^{-1}]) \end{array}$$

The function $f_A : G_{n,n+k}(\mathbb{C}[z, z^{-1}]) \to \mathbb{R}$ is the direct limit of a system of Morse-Bott-Smale functions defined on finite dimensional Grassmann manifolds consisting of n-planes inside the $(n+k)$-product of the finite dimensional complex vector space of Laurent polynomials whose degrees are bounded by some integer $j \in \mathbb{Z}_+$.

For each $j \in \mathbb{Z}_+$ define $\mathbb{C}[z, z^{-1}]_j$ to be the collection of all Laurent polynomials of the form

$$a_{-j}z^{-j} + a_{-j+1}z^{-j+1} + \cdots + a_{j-1}z^{j-1} + a_j z^j.$$

We have a smooth action

$$GL_n(\mathbb{C}) \times V_{n,n+k}(\mathbb{C}[z, z^{-1}]_j) \to V_{n,n+k}(\mathbb{C}[z, z^{-1}]_j)$$

given by matrix multiplication on the left by an element of $GL_n(\mathbb{C})$ (see for instance [10] p. 193-4 or [9] p. 94-5) and the quotient space is

$$V_{n,n+k}(\mathbb{C}[z, z^{-1}]_j)/GL_n(\mathbb{C}) = G_{n,n+k}(\mathbb{C}[z, z^{-1}]_j) \approx G_{n,(n+k)(2j+1)}(\mathbb{C})$$

the Grassmann manifold of n-planes in $\mathbb{C}[z, z^{-1}]_j^{n+k}$. Taking a direct limit over j we have the infinite dimensional complex Grassmann manifold

$$V_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n(\mathbb{C}) = G_{n,n+k}(\mathbb{C}[z, z^{-1}]) \approx G_n(\mathbb{C}^\infty).$$

$G_{n,n+k}(\mathbb{C}[z, z^{-1}]_j)$ is diffeomorphic to the orbit of the adjoint action of the unitary group $U_{(n+k)(2j+1)}$ whose spectrum is $(1, \ldots, 1) \times (0, \ldots, 0) \in \mathbb{R}^n \times \mathbb{R}^{(n+k)2j+k}$ (see for instance [1] p. 54-55). Choosing a diagonal matrix in this orbit, x_0, we have defined a unique $U_{(n+k)(2j+1)}$-equivariant diffeomorphism

$$\phi : G_{n,n+k}(\mathbb{C}[z, z^{-1}]) \to U_{(n+k)(2j+1)} \cdot x_0.$$

For every $j \in \mathbb{Z}_+$ let M_j be the $(2j+1) \times (2j+1)$ diagonal matrix whose mth diagonal entry is $m\sqrt{-1}$ where $-j \leq m \leq j$ (we are indexing the entries of M_j by $\{-j,-j+1,\ldots,j-1,j\}$). Let

$$A_j = \begin{pmatrix} M_j & & & 0 \\ & M_j & & \\ & & \ddots & \\ 0 & & & M_j \end{pmatrix} \in u((n+k)(2j+1))$$

be the skew-Hermitian diagonal matrix with $n+k$ blocks of M_j along the diagonal. These matrices define Morse-Bott-Smale functions $f_{A_j} : G_{n,n+k}(\mathbb{C}[z,z^{-1}]_j) \to \mathbb{R}$ given by $f_{A_j}(p) = <\phi(p), A_j>$ for all $p \in G_{n,n+k}(\mathbb{C}[z,z^{-1}]_j)$ where $<\cdot,\cdot>$ denotes the Killing form .

For every $p \in G_{n,n+k}(\mathbb{C}[z,z^{-1}]_j)$ the gradient flow line of f_{A_j} through p, with respect to the pullback under ϕ of the Killing form, is $\gamma_p(t) = L_t(p)$ where $L_t : \mathbb{C}[z,z^{-1}]_j^{n+k} \to \mathbb{C}[z,z^{-1}]_j^{n+k}$ is the linear map determined by the matrix $\exp(-itA_j)$ [12]. This flow lifts to $V_{n,n+k}(\mathbb{C}[z,z^{-1}]_j)$ because $\exp(-itA_j)$ acts on $V_{n,n+k}(\mathbb{C}[z,z^{-1}]_j)$ by matrix multiplication on the right and $GL_n(\mathbb{C})$ acts by matrix multiplication on the left. If $M \in V_{n,n+k}(\mathbb{C}[z,z^{-1}]_j)$ has $l_{ij}(z) \in \mathbb{C}[z,z^{-1}]$ in its (i,j) entry, then the matrix $M\exp(-itA_j) \in V_{n,n+k}(\mathbb{C}[z,z^{-1}]_j)$ has $l_{ij}(e^t z) \in \mathbb{C}[z,z^{-1}]$ as its (i,j) entry. That is, the gradient flow of f_{A_j} is given by composing $l_{ij}(z)$ with the map $z \mapsto e^t z$ for all $i,j \in \mathbb{Z}$. The function $f_A : G_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathbb{R}$ referred to in Theorem 9 is defined by $f_A = \lim_j f_{A_j}$. The gradient flow lines of f_A are given by composing each Laurent polynomial entry of an element of $V_{n,n+k}(\mathbb{C}[z,z^{-1}])$ with the map $z \mapsto e^t z$.

Claim 10. *$P_{n,n+k}(\mathbb{C}[z,z^{-1}])$ is an open dense subset of $V_{n,n+k}(\mathbb{C}[z,z^{-1}])$.*

Proof:

$M \in V_{n,n+k}(\mathbb{C}[z,z^{-1}])$ is in $P_{n,n+k}(\mathbb{C}[z,z^{-1}])$ if and only if for every $w \in \mathbb{C}^*$ the determinant of at least one $n\times n$ minor of $M(w)$ is non-zero. The determinants of the $n \times n$ minors of M are Laurent polynomials and hence have only finitely many roots. By perturbing the entries of M slightly we can insure that these $\binom{n+k}{n}$ polynomials do not have a root in common. This shows that $P_{n,n+k}(\mathbb{C}[z,z^{-1}])$ is dense in $V_{n,n+k}(\mathbb{C}[z,z^{-1}])$. If $M \in P_{n,n+k}(\mathbb{C}[z,z^{-1}])$ then the determinants of the $n\times n$ minors of M do not have a root in common. If we perturb the entries of M slightly then the determinants of the $n \times n$ minors of the perturbed matrix won't have a root in common either. This shows that $P_{n,n+k}(\mathbb{C}[z,z^{-1}])$ is open in $V_{n,n+k}(\mathbb{C}[z,z^{-1}])$.

□

$GL_n(\mathbb{C}[z,z^{-1}])$ acts on the left of $M_{n,n+k}(\mathbb{C}[z,z^{-1}])$ by matrix multiplication. This action does not restrict to $V_{n,n+k}(\mathbb{C}[z,z^{-1}])$, but it does restrict to $P_{n,n+k}(\mathbb{C}[z,z^{-1}])$. The proof of the following claim is similar to that of the preceding.

Claim 11. *There exists an action*

$$GL_n(\mathbb{C}[z,z^{-1}]) \times P_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to P_{n,n+k}(\mathbb{C}[z,z^{-1}])$$

given by matrix multiplication on the left by an element of $GL_n(\mathbb{C}[z,z^{-1}])$.

Note that this action corresponds to Laurent polynomial row operations on an element of $P_{n,n+k}(\mathbb{C}[z,z^{-1}])$. That is, by multiplying an element of $P_{n,n+k}(\mathbb{C}[z,z^{-1}])$ on the left by an element of $GL_n(\mathbb{C}[z,z^{-1}])$ we can interchange rows, multiply a row by a unit of $\mathbb{C}[z,z^{-1}]$, or add a Laurent polynomial multiple of one row to another.

$GL_n^c(\mathbb{C}[z,z^{-1}])$ is the kernel of the homomorphism $GL_n(\mathbb{C}[z,z^{-1}]) \to \mathbb{Z}$ which sends a matrix to the degree of its determinant. Hence, $GL_n^c(\mathbb{C}[z,z^{-1}])$ is a normal subgroup of $GL_n(\mathbb{C}[z,z^{-1}])$, the quotient group $GL_n(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}]) = \mathbb{Z}$, and the restriction

$$GL_n^c(\mathbb{C}[z,z^{-1}]) \times P_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to P_{n,n+k}(\mathbb{C}[z,z^{-1}])$$

of the $GL_n(\mathbb{C}[z,z^{-1}])$ action is free and gives the following fiber bundle

$$\begin{array}{ccc} GL_n^c(\mathbb{C}[z,z^{-1}])/GL_n(\mathbb{C}) & \longrightarrow & P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n(\mathbb{C}) \\ & & \Big\downarrow \pi \\ & & P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}]). \end{array}$$

The reader should note that $GL_n(\mathbb{C})$ is **not** a normal subgroup of $GL_n^c(\mathbb{C}[z,z^{-1}])$. The following lemma is an immediate consequence of the fact that the gradient flow of $f_A : G_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathbb{R}$ is given by composing Laurent polynomials with the map $z \mapsto e^t z$.

Lemma 12. *The gradient flow of* $f_A : G_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathbb{R}$ *restricts to a flow on* $P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n(\mathbb{C})$. *This flow descends to a flow on* $P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}])$ *such that* π *is* $\mathbb{R}$*-equivariant.*

Let $\mathcal{C}_{n,n+k}$ be the flow category on $P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}])$ induced from the gradient flow of $f_A : G_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathbb{R}$. Theorem 9 asserts that $\mathcal{C}_{n,n+k}$ is isomorphic to $\mathcal{C}_{\tilde{\mathcal{A}}_{n,n+k}}$, the flow category of the action functional $\tilde{\mathcal{A}} : \widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C}) \to \mathbb{R}$. Before giving a rigorous proof of the theorem we give the following heuristic argument.

Recall that $P_{n,n+k}(\mathbb{C}[z,z^{-1}])$ is the space of all Laurent polynomial maps $\mathbb{C}^* \to V_{n,n+k}(\mathbb{C})$. We can define a continuous injective map

$$P_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathcal{L}V_{n,n+k}(\mathbb{C})$$

into the space of all continuous maps $S^1 \to V_{n,n+k}(\mathbb{C})$ by restricting an element of $P_{n,n+k}(\mathbb{C}[z,z^{-1}])$ to S^1. Similarly, $GL_n^c(\mathbb{C}[z,z^{-1}])$ maps into the identity component, $\mathcal{L}_0 GL_n$, of the space of all continuous maps $S^1 \to GL_n(\mathbb{C})$ and the map

$$P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}]) \to \mathcal{L}V_{n,n+k}(\mathbb{C})/\mathcal{L}_0 GL_n = \widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})$$

is continuous, injective, and surjects onto the space of algebraic points in $\widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})$. This map is $\mathbb{R}$-equivariant with respect to the flow induced from $f_A : G_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathbb{R}$ on the left and the flow of the action functional on the right.

If the above map was a homeomorphism onto the space of algebraic loops, then we would have an induced isomorphism of flow categories

$$\mathcal{C}_{n,n+k} \to \mathcal{C}_{\tilde{\mathcal{A}}_{n,n+k}}$$

by Theorem 6. However, the above map is definitely **not** a homeomorphism onto the space of algebraic loops as can be seen even in the simple case $n = k = 1$ studied by Cohen, Jones, and Segal in [5]. For example one can find sequences in $P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}])$ that do not converge but whose images do converge to algebraic loops. Even though the above map is not a homeomorphism onto its image, it does induce an isomorphism of flow categories. This is possible because the morphisms in the flow category are "lines on the manifold" rather than individual points in the manifold.

Proof of Theorem 9:

Let $N = \binom{n+k}{n}$. The Plücker embedding $Pl : G_{n,n+k}(\mathbb{C}) \to \mathbb{C}P^{N-1}$ is defined by sending a plane to the homogeneous coordinates given by the determinants of the $n \times n$ minors of any element of $V_{n,n+k}(\mathbb{C})$ whose rows span the plane. We have a similar map

$$Pl : P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}]) \to \mathbb{P}(\mathbb{C}[z,z^{-1}]^N)$$

defined by sending an equivalence class $[M]$ to the N-tuple of Laurent polynomials (mod $\mathbb{C}^*$) given by the determinants of the $n \times n$ minors of M. This generalized Plücker embedding is well-defined because multiplying M by an element of $GL_n^c(\mathbb{C}[z,z^{-1}])$ can only change the determinants of the $n \times n$ minors of M by an element of $\mathbb{C}^*$.

Lemma 13. *$Pl : P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}]) \to \mathbb{P}(\mathbb{C}[z,z^{-1}]^N)$ is an embedding.*

Proof:

Assume that the determinants of the $n \times n$ minors of $M_1, M_2 \in P_{n,n+k}(\mathbb{C}[z, z^{-1}])$ are the same up to multiplication by an element of $\mathbb{C}^*$. Since the standard Plücker embedding $G_{n,n+k}(\mathbb{C}) \to \mathbb{C}P^{N-1}$ is injective, there exists a matrix of functions $G(z) = (g_{ij}(z))$ (i.e. $g_{ij} : \mathbb{C}^* \to \mathbb{C}$ for all $1 \leq i, j \leq n$) such that $G(w)M_1(w) = M_2(w)$ for all $w \in \mathbb{C}^*$. Since $M_1 \in P_{n,n+k}(\mathbb{C}[z, z^{-1}])$ there exists a minor of M_1, say $(M_1)_I$, whose determinant is not the zero polynomial. For every $1 \leq j \leq n$ the jth row of G gives a system of n equations and n unknowns in $g_{1j}, g_{2j}, \ldots, g_{nj}$

$$(g_{j1}, g_{j2}, \ldots, g_{jn})(M_1)_I = (l_{j1}, l_{j2}, \ldots, l_{jn})$$

where $l_{j1}, l_{j2}, \ldots, l_{jn}$ are the entries in the jth row of the minor $(M_2)_I$. If we multiply both sides of the above equations by a high enough power of z to clear the negative powers of the Laurent polynomials, then we have an equivalent system of equations in the sense that the functions g_{ij} which solve one system also solve the other. This new system of equations is a linear system of n equations and n unknowns over the field of rational functions. Moreover, since the determinant of $(M_1)_I$ is not zero this new system of equations has a solution over the field of rational functions. That is, the functions g_{ij} are rational functions that can only have poles at zero. So as functions the g_{ij} are Laurent polynomials. Hence, $G \in GL_n^c(\mathbb{C}[z, z^{-1}])$. This shows that $Pl : P_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n^c(\mathbb{C}[z, z^{-1}]) \to \mathbb{P}(\mathbb{C}[z, z^{-1}]^N)$ is injective.

It's clear that $Pl : P_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n^c(\mathbb{C}[z, z^{-1}]) \to \mathbb{P}(\mathbb{C}[z, z^{-1}]^N)$ is continuous. To see that the inverse map is continuous it suffices to show that the composite

$$P_{n,n+k}(\mathbb{C}[z, z^{-1}]) \xrightarrow{\pi} P_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n^c(\mathbb{C}[z, z^{-1}]) \xrightarrow{Pl}$$

$$Pl(P_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n^c(\mathbb{C}[z, z^{-1}])) \subseteq \mathbb{P}(\mathbb{C}[z, z^{-1}]^N)$$

maps open sets to open sets. But every point $M \in P_{n,n+k}(\mathbb{C}[z, z^{-1}])$ has an open neighborhood given by perturbing the coefficients of the entries of M which maps onto an open neighborhood of $Pl \circ \pi(M)$. That is, $Pl \circ \pi(M) \in \mathbb{P}(\mathbb{C}[z, z^{-1}]^N)$ has homogeneous coordinates which are linear functions in the coefficients of the Laurent polynomial entries of M and since a linear function of several variables is an open map $Pl \circ \pi$ is an open map.

□

The map $Pl : P_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n^c(\mathbb{C}[z, z^{-1}]) \to \mathbb{P}(\mathbb{C}[z, z^{-1}]^N)$ is $\mathbb{R}$-equivariant, i.e. both the flow on $P_{n,n+k}(\mathbb{C}[z, z^{-1}])/GL_n^c(\mathbb{C}[z, z^{-1}])$

described in Lemma 12 and the flow of $f_A : \mathbb{P}(\mathbb{C}[z,z^{-1}]^N) \to \mathbb{R}$ restricted to the image of the generalized Plücker embedding are given by composing Laurent polynomials with the map $z \mapsto e^t z$. Hence by Theorem 6 the flow category on $P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}])$ is isomorphic to the subcategory of $f_A : \mathbb{P}(\mathbb{C}[z,z^{-1}]^N) \to \mathbb{R}$ consisting of those critical points and flow lines which lie in the image of Pl. Note that the object space of $\mathcal{C}_{n,n+k}$ is homeomorphic to $\mathbb{Z} \times G_{n,n+k}(\mathbb{C})$.

Every point $[M] \in P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}])$ determines a unique algebraic point in $\widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})$ as follows. Recall that a point in $\widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})$ is given by a map $S^1 \to G_{n,n+k}(\mathbb{C})$ together with an extension $D^2 \to G_{n,n+k}(\mathbb{C})$ well defined up to homotopy rel S^1. First we label the critical submanifold in $\widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})_{\text{alg}}$ consisting of constant extensions of constant loops to D^2 by C_0. The other critical submanifolds are then labeled in relation to C_0, i.e. the critical submanifold on sheet $j \in \mathbb{Z}$ of the universal cover is labeled C_j. The preceding lemma implies that every element of $P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}])$ determines a unique holomorphic map $\mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ and hence a gradient flow line of the action functional from a constant loop at some point $a \in G_{n,n+k}(\mathbb{C})$ to a constant loop at some $b \in G_{n,n+k}(\mathbb{C})$. We lift this flow to a map $\mathbb{R} \to \widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})$ beginning at $a \in C_j = G_{n,n+k}(\mathbb{C})$ and then evaluate at zero where j is the unique element of $\mathbb{Z}$ such that multiplying each entry in the N-tuple $Pl([M])$ by z^{-j} gives a collection of elements of $\mathbb{C}[z]$ with no common roots in $\mathbb{C}$. This defines a continuous bijective map

$$i_{n,n+k} : P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}]) \to \widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})_{\text{alg}}.$$

It's clear from the definition that this map is $\mathbb{R}$-equivariant with respect to the induced flow from $f_A : G_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathbb{R}$ on the left and the gradient flow of the action functional on the right, i.e. on both sides the flow is given by $z \mapsto e^t z$. Therefore by Theorem 6 $i_{n,n+k}$ induces a continuous bijective functor

$$I_{n,n+k} : \mathcal{C}_{n,n+k} \to \mathcal{C}_{\tilde{\mathcal{A}}_{n,n+k}}.$$

It is clear that $I_{n,n+k} : Ob(\mathcal{C}_{n,n+k}) \to Ob(\mathcal{C}_{\tilde{\mathcal{A}}_{n,n+k}})$ is a homeomorphism because $i_{n,n+k}$ is continuous and bijective and the object space of $\mathcal{C}_{n,n+k}$ has compact components (i.e. each component is $G_{n,n+k}(\mathbb{C})$).

The Plücker embedding $Pl : G_{n,n+k}(\mathbb{C}) \to \mathbb{C}P^{N-1}$ also induces an embedding $\mathcal{L}Pl : \mathcal{L}G_{n,n+k}(\mathbb{C}) \to \mathcal{L}\mathbb{C}P^{N-1}$ and since $\pi_1(\mathcal{L}G_{n,n+k}(\mathbb{C})) = \pi_1(\mathcal{L}\mathbb{C}P^{N-1}) = \mathbb{Z}$ this induces an embedding

$$\widetilde{\mathcal{L}Pl} : \widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})) \to \widetilde{\mathcal{L}\mathbb{C}P^{N-1}}.$$

Chasing through the definitions of the maps involved one sees that we have the following $\mathbb{R}$-equivariant commutative diagram.

$$\begin{array}{ccc} P_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n^c(\mathbb{C}[z,z^{-1}]) & \xrightarrow{i_{n,n+k}} & \widetilde{\mathcal{L}G}_{n,n+k}(\mathbb{C})_{\text{alg}} \\ \downarrow{\scriptstyle Pl} & & \downarrow{\scriptstyle \widetilde{\mathcal{L}Pl}} \\ P_{1,N}(\mathbb{C}[z,z^{-1}])/\mathbb{C}^* & \xrightarrow{i_{1,N}} & \widetilde{\mathcal{L}\mathbb{C}P^{N-1}}_{\text{alg}} \end{array}$$

The preceding diagram induces the following commutative diagram of flow categories by Theorem 6.

$$\begin{array}{ccc} \mathcal{C}_{n,n+k} & \xrightarrow{I_{n,n+k}} & \mathcal{C}_{\tilde{\mathcal{A}}_{n,n+k}} \\ \downarrow & & \downarrow \\ \mathcal{C}_{1,N} & \xrightarrow{I_{1,N}} & \mathcal{C}_{\tilde{\mathcal{A}}_{1,N}} \end{array}$$

where the vertical arrows are inclusion functors induced by the Plücker embeddings in the preceding diagram.

One of the results announced in [5] is that the functor $I_{1,N} : \mathcal{C}_{1,N} \to \mathcal{C}_{\tilde{\mathcal{A}}_{1,N}}$ is an isomorphism of categories. In particular, for any $a, b \in Ob(\mathcal{C}_{n,n+k})$ we have a homeomorphism

$$I_{1,N} : \text{Mor}(a,b)_{\mathcal{C}_{1,N}} \to \text{Mor}(i_{1,N}(a), i_{1,N}(b))_{\mathcal{C}_{\tilde{\mathcal{A}}_{1,N}}}$$

where $\text{Mor}(a,b)_{\mathcal{C}_{1,N}}$ is a morphism space in the flow category of the function $f_A : \mathbb{P}(\mathbb{C}[z,z^{-1}]^N) \to \mathbb{R}$ and $\text{Mor}(i_{1,N}(a), i_{1,N}(b))_{\mathcal{C}_{\tilde{\mathcal{A}}_{1,N}}}$ is a morphism space in the flow category of the action functional on $\widetilde{\mathcal{L}\mathbb{C}P^{N-1}}$.

$$I_{n,n+k} : \text{Mor}(a,b)_{\mathcal{C}_{n,n+k}} \to \text{Mor}(i_{n,n+k}(a), i_{n,n+k}(b))_{\mathcal{C}_{\tilde{\mathcal{A}}_{n,n+k}}}$$

is simply a restriction of $I_{1,N}$ and hence is a homeomorphism. Therefore $I_{n,n+k} : \mathcal{C}_{n,n+k} \to \mathcal{C}_{\tilde{\mathcal{A}}_{n,n+k}}$ is an isomorphism of flow categories.

□

One should note that our definition of $\text{Mor}(a,b)$ as the space of piecewise flow lines on the manifold immediately gives the result of Cohen, Jones, and Segal announced in [5]. We have seen that

$$i_{1,N} : P_{1,N}(\mathbb{C}[z,z^{-1}])/\mathbb{C}^* \to \widetilde{\mathcal{L}\mathbb{C}P^{N-1}}_{\text{alg}}$$

is a continuous bijective map whose inverse is discontinuous. However, restricted to the space of critical submanifolds this map is a homeomorphism. Hence by Theorem 6 there is an induced continuous bijective functor

$$I_{1,N} : \mathcal{C}_{1,N} \to \mathcal{C}_{\tilde{\mathcal{A}}_{1,N}}$$

that is a homeomorphism on the object spaces.

To see that the inverse map on the morphism spaces is continuous fix any two critical points $a, b \in Ob(\mathcal{C}_{\tilde{\mathcal{A}}_{1,N}})$ and assume $l_j \in \mathrm{Mor}(a,b)_{\mathcal{C}_{\tilde{\mathcal{A}}_{1,N}}}$ is a sequence of piecewise gradient flow lines converging to $l \in \mathrm{Mor}(a,b)_{\mathcal{C}_{\tilde{\mathcal{A}}_{1,N}}}$. If $I_{1,N}^{-1}(l_j)$ did not converge to $I_{1,N}^{-1}(l)$, then there would be a sequence of points $p_j \in I_{1,N}^{-1}(l_j)$ that stayed some finite distance from $I_{1,N}^{-1}(l)$. But by assumption $I_{1,N}(p_j)$ approaches l and since l is compact there is a subsequence of $I_{1,N}(p_j)$ approaching some point $I_{1,N}(p) \in l$. Hence to show that $I_{1,N}^{-1}(l_j)$ converges to $I_{1,N}^{-1}(l)$ it suffices to show that for every $I_{1,N}(p) \in l$ and every sequence of points $I_{1,N}(p_j) \in l_j$ converging to $I_{1,N}(p)$, p_j converges to p.

Pick any $I_{1,N}(p) \in l$ and assume that $I_{1,N}(p_j) \in l_j$ is a sequence converging to $I_{1,N}(p)$. Applying the projection map

$$\pi : \widetilde{\mathcal{L}\mathbb{C}P^{N-1}} \to \mathcal{L}\mathbb{C}P^{N-1}$$

we have a sequence $\pi(I_{1,N}(p_j))$ converging to $\pi(I_{1,N}(p))$. Since all these points are algebraic there exist holomorphic maps

$$h_j : \mathbb{C}P^1 \to \mathbb{C}P^{N-1}$$

and

$$h : \mathbb{C}P^1 \to \mathbb{C}P^{N-1}$$

such that $h_j(e^{is}) = \pi(I_{1,N}(p_j))$ and $h(e^{is}) = \pi(I_{1,N}(p))$. These holomorphic maps are specific parameterizations for segments of the piecewise gradient flow lines $\pi(l_j)$ and $\pi(l)$ and because of the way the parameterizations were chosen h_j approaches h on S^1 (the image of e^{is}) as $j \to \infty$. In other words, h is a "bubble" in the limit of h_j where the bubbling can only occur at 0 and ∞ because the $\pi(l_j)$ are all piecewise gradient flow lines of the action functional. Since these maps are holomorphic $h_j(z) = (p_j^1(z) : \cdots : p_j^N(z))$ for every $j \in \mathbb{Z}_+$ and $h(z) = (p^1(z) : \cdots : p^N(z))$ where the entries are polynomials with no root in common in $\mathbb{C}$. As elements of $\mathbb{P}(\mathbb{C}[z, z^{-1}]^N)$

$$(p_j^1(z) : \cdots : p_j^N(z)) \to z^m(p^1(z) : \cdots : p^N(z))$$

as $j \to \infty$ where $m \in \mathbb{Z}$ is determined by what sort of bubbling occurs. After multiplying by an appropriate power of z, determined by what sheet in the universal cover $I_{1,N}(p)$ lies, we have p_j and p. Hence $p_j \to p$ as $j \to \infty$.

4. Floer Homotopy Type and Floer Cohomology

Definition 14. *Fixing any critical submanifold $C_0 \subseteq Ob(\mathcal{C}_{f_A})$ we define the Floer homotopy type of $f_A : G_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathbb{R}$ (following* [3]*) to be*

$$\varinjlim_{B\in I}\{(B\mathcal{C}_{f_A}|_B^{C_0})^{-\nu_C}\}_{C\in I_B}$$

where

$$I = \{B \text{ a component of } Ob(\mathcal{C}_{f_A}) | \overline{W(C_0,B)} \text{ is a manifold}\}$$

and

$$I_B = \{C \text{ a component of } Ob(\mathcal{C}_{f_A}) | C_0 \leq C \text{ and } \overline{W(C,B)} \text{ is a manifold}\}$$

and ν_C is the normal bundle of

$$B\mathcal{C}_{f_A}|_B^{C_0} \overset{\nu_C}{\hookrightarrow} B\mathcal{C}_{f_A}|_B^{C}.$$

The maps in the above systems are all inclusion maps.

Lemma 15. *$P \in G_{n,n+k}(\mathbb{C}[z,z^{-1}]) = V_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n(\mathbb{C})$ is a critical point of $f_A : G_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathbb{R}$ if and only if there exists some $M \in V_{n,n+k}(\mathbb{C}[z,z^{-1}])$ lying above P such that the ith row of M consists of entries of the form $a_{ij}z^{m_i}$ for some $a_{ij} \in \mathbb{C}$ and some $m_i \in \mathbb{Z}$.*

Proof:
The critical points of $f_A : G_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathbb{R}$ are the fixed points of the gradient flow. Recall that this gradient flow is given by composing the Laurent polynomial entries of $M \in V_{n,n+k}(\mathbb{C}[z,z^{-1}])$ with the function $z \mapsto e^t z$. Assume that $M \in V_{n,n+k}(\mathbb{C}[z,z^{-1}])$ is of the form described above. Then composing the entries of M with $z \mapsto e^t z$ multiplies the ith row by $e^{m_i t}$. Hence, the equivalence class does not change in $V_{n,n+k}(\mathbb{C}[z,z^{-1}])/GL_n(\mathbb{C}) = G_{n,n+k}(\mathbb{C}[z,z^{-1}])$.

Now assume that $P \in G_{n,n+k}(\mathbb{C}[z,z^{-1}])$ is fixed point. Choose $M \in V_{n,n+k}(\mathbb{C}[z,z^{-1}])$ lying above P and in reduced row echelon form. Since the rows of $M(e^t z)$ span the same plane as the rows of $M(z)$ the first row of $M(e^t z)$ is a multiple of the first row of $M(z)$. Thus the first row of $M(z)$ must consist of entries of the form $a_{1j}z^{m_1}$ for some $m_1 \in \mathbb{Z}$. By repeating the argument for rows $2, 3, \ldots, n$ we see that every row of M must consist of entries of the form $a_{ij}z^{m_i}$ for some $m_i \in \mathbb{Z}$.

□

The preceding lemma shows that the critical submanifolds of $f_A : G_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathbb{R}$ are indexed by n-tuples of integers $(m_1,\dots,m_n)$. We will denote these critical submanifolds by $C_{(m_1,\dots,m_n)}$. If $m_1 = m_2 = \cdots = m_n$, then $C_{(m_1,\dots,m_n)} = G_{n,n+k}(\mathbb{C})$. In general, $C_{(m_1,\dots,m_n)}$ is a product of Grassmann manifolds, e.g. if l of the integers $(m_1,\dots,m_n)$ are the same, then $C_{(m_1,\dots,m_n)}$ will have $G_{l,n+k}(\mathbb{C})$ as a factor. From this point on we will fix $C_0 = C_{(0,0,\dots,0)}$.

Theorem 16. *The Floer homotopy type of $f_A : G_{n,n+k}(\mathbb{C}[z,z^{-1}]) \to \mathbb{R}$ has*

$$G_n(\mathbb{C}^\infty) \leftarrow G_n(\mathbb{C}^\infty)^{-(n+k)\gamma_n} \leftarrow G_n(\mathbb{C}^\infty)^{-2(n+k)\gamma_n} \leftarrow \cdots$$

as a cofinal system where γ_n denotes the tautological n-plane bundle over $G_n(\mathbb{C}^\infty)$.

Proof:
Let $C_j = C_{(j,j,\dots,j)}$ for all $j \in \mathbb{Z}$. Then for $j < 0$ we have $B\mathcal{C}_{f_A}|_{C_j}^{C_0} \cong \overline{W(C_0,C_j)} = G_{n,n+k}(\mathbb{C}[z,z^{-1}]^{\{j,j+1,\dots,0\}})$ where $\mathbb{C}[z,z^{-1}]^{\{j,j+1,\dots,0\}}$ denotes the collection of all Laurent polynomials of the form

$$a_j z^j + a_{j+1}z^{j+1} + \cdots + a_{-1}z^{-1} + a_0.$$

The normal bundle of the embedding $B\mathcal{C}_{f_A}|_{C_j}^{C_0} \hookrightarrow B\mathcal{C}_{f_A}|_{C_j}^{C_m}$ is $m(n+k)\gamma_n$ for all $m \in \mathbb{Z}_+$. Hence $\{(B\mathcal{C}_{f_A}|_{C_j}^{C_0})^{-\nu_C}\}_{C \in I_{C_j}}$ has

$$G_n(\mathbb{C}[z,z^{-1}]^{\{j,j+1,\dots,0\}}) \leftarrow G_n(\mathbb{C}[z,z^{-1}]^{\{j,j+1,\dots,0\}})^{-(n+k)\gamma_n} \leftarrow \cdots$$

as a cofinal system.

Taking a direct limit $j \to -\infty$ we see that

$$G_n(\mathbb{C}^\infty) \leftarrow G_n(\mathbb{C}^\infty)^{-(n+k)\gamma_n} \leftarrow G_n(\mathbb{C}^\infty)^{-2(n+k)\gamma_n} \leftarrow \cdots$$

is a cofinal system of the Floer homotopy type.

□

If we apply H^* to the above pro-spectrum and use the Thom Isomorphism Theorem we get the following direct system.

$$H^*(G_n(\mathbb{C}^\infty)) \overset{\cup c_n^{n+k}}{\to} H^{*+2n(n+k)}(G_n(\mathbb{C}^\infty)) \overset{\cup c_n^{n+k}}{\to} \cdots$$

The direct limit of this system is

$$H^*(G_n(\mathbb{C}^\infty))[c_n^{-(n+k)}].$$

The reader should note that the above cohomology groups are graded over $\mathbb{Z}$ and periodic with period $2(n+k)$ where $n+k$ is the minimal Chern number of $G_{n,n+k}(\mathbb{C})$. This is consistent with the grading on Floer's cohomology groups.

For the case $n = 1$ Theorem 9 implies that the flow category $\mathcal{C}_{f_A}$ is a compactification of the flow category of the action functional on the universal cover of $\mathcal{L}\mathbb{C}P^k$. We define the Floer homotopy type of the action functional to be the Floer homotopy type of $\mathcal{C}_{f_A}$. Looking at the proof of Theorem 16 we see that the Floer homotopy type of the action functional on the universal cover of $\mathcal{L}\mathbb{C}P^k$ is

$$\mathbb{C}P^\infty \leftarrow (\mathbb{C}P^\infty)^{-(1+k)\gamma_1} \leftarrow (\mathbb{C}P^\infty)^{-2(1+k)\gamma_1} \leftarrow \cdots$$

where γ_1 denotes the Hopf line bundle over $\mathbb{C}P^\infty$.

The Floer cohomology ring of $\mathbb{C}P^k$ is well-known [14].

Theorem 17. *The Floer cohomology ring of* $\mathbb{C}P^k$ *is*

$$HF^*(\mathbb{C}P^k) = \frac{\mathbb{Z}[p,q,q^{-1}]}{< p^{1+k} = q, q^{-1}q = 1 >}$$

where p *has degree 2.*

Note that $HF^*(\mathbb{C}P^k)$ is isomorphic to

$$H^*(\mathbb{C}P^\infty)[(c_1)^{-(1+k)}] \cong \mathbb{Z}[c_1, c_1^{-(1+k)}]$$

One reason that $HF^*(\mathbb{C}P^k)$ is usually written as in the above theorem is to stress the $2N$-periodicity of the Floer cohomology groups where $N = 1 + k$ is the minimal Chern number of $\mathbb{C}P^k$. Another reason is that it exhibits the action of $\pi_2(\mathbb{C}P^k)$ on $HF^*(\mathbb{C}P^k)$.

$\pi_2(\mathbb{C}P^k) = \pi_1(\mathcal{L}\mathbb{C}P^k)$ is the group of deck transformations of $\widetilde{\mathcal{L}\mathbb{C}P}^k$. This action induces an action on the Floer chain complex. $A \in \pi_2(\mathbb{C}P^k)$ sends a critical point of Maslov index μ to a critical point of Maslov index $\mu + 2c_1(A)$ (see [14] section 10.1). If A is a generator of $\pi_2(\mathbb{C}P^k)$, then the induced action of A on the Floer cohomology ring is multiplication by either q or q^{-1} in Theorem 17.

The group of deck transformations $\pi_2(\mathbb{C}P^k)$ sends critical points to critical points and gradient flow lines to gradient flow lines. Thus there is an induced action of $\pi_2(\mathbb{C}P^k)$ on the flow category of the action functional on the universal cover of $\mathcal{L}\mathbb{C}P^k$. This action has the effect of reindexing the critical submanifolds in Definition 14, but other than that it does not change the Floer homotopy type. In particular, we see that the Floer homotopy type of the action functional on the universal cover of $\mathcal{L}\mathbb{C}P^k$ is independent of the basepoint chosen in Definition 14.

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