THE SYMPLECTIC DISPLACEMENT ENERGY

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Abstract. We define the symplectic displacement energy of a non-empty subset of a compact symplectic manifold as the infimum of the Hofer-like norm [5] of symplectic diffeomorphisms that displace the set. We show that this energy (like the usual displacement energy defined using Hamiltonian diffeomorphisms) is a strictly positive number on sets with non-empty interior. As a consequence we prove a result justifying the introduction of the notion of strong symplectic homeomorphisms [4].

1. Statement of results

In [14], Hofer defined a norm $\|\cdot\|_H$ on the group $\mathrm{Ham}(M,\omega)$ of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold (M,ω).

For a non-empty subset $A \subset M$, he introduced the notion of the **displacement energy** $e(A)$ of A:

$$e(A) = \inf\{\|\phi\|_H \mid \phi \in \mathrm{Ham}(M,\omega), \phi(A) \cap A = \emptyset\}.$$

The displacement energy is defined to be $+\infty$ if no compactly supported Hamiltonian diffeomorphism displaces A.

Eliashberg and Polterovich [8] proved the following result.

Theorem 1. *For any non-empty open subset A of M, $e(A)$ is a strictly positive number.*

It is easy to see that if A and B are non-empty subsets of M such that $A \subset B$, then $e(A) \leq e(B)$, and that e is a symplectic invariant. That is,

$$e(f(A)) = e(A)$$

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for all $f \in \mathrm{Symp}(M,\omega) = \{\phi \in \mathrm{Diff}(M) \mid \phi^*\omega = \omega\}$. This follows from the fact that $\|f \circ \phi \circ f^{-1}\|_H = \|\phi\|_H$.

In [5], a Hofer-like metric $\|\cdot\|_{HL}$ was constructed on the group $\mathrm{Symp}_0(M,\omega)$ of all symplectic diffeomorphisms of a compact symplectic manifold (M,ω) that are isotopic to the identity. It was proved recently by Buss and Leclercq [7] that the restriction of $\|\cdot\|_{HL}$ to $\mathrm{Ham}(M,\omega)$ is a metric equivalent to the Hofer metric.

Let us now propose the following definition.

Definition 2. The **symplectic displacement energy** $e_s(A)$ of a non-empty subset $A \subset M$ is defined to be:

$$e_s(A) = \inf\{\|h\|_{HL} \mid h \in \mathrm{Symp}_0(M,\omega), h(A) \cap A = \emptyset\}$$

if some element of $\mathrm{Symp}_0(M,\omega)$ displaces A, and $+\infty$ if no element of $\mathrm{Symp}_0(M,\omega)$ displaces A.

Clearly, if A and B are non-empty subsets of M such that $A \subset B$, then $e_s(A) \leq e_s(B)$.

The goal of this paper is to prove the following result.

Theorem 3. *For any closed symplectic manifold (M,ω), the symplectic displacement energy of any subset $A \subset M$ with non-empty interior satisfies $e_s(A) > 0$.*

2. The Hofer norm $\|\cdot\|_H$ and the Hofer-like norm $\|\cdot\|_{HL}$

2.1. **$\mathbf{Symp_0(M,\omega)}$ and $\mathbf{Ham(M,\omega)}$.** Let $\mathrm{Iso}(M,\omega)$ be the set of all compactly supported symplectic isotopies of a symplectic manifold (M,ω). A compactly supported symplectic isotopy $\Phi \in \mathrm{Iso}(M,\omega)$ is a smooth map $\Phi : M \times [0,1] \to M$ such that for all $t \in [0,1]$, if we denote by $\phi_t(x) = \Phi(x,t)$, then $\phi_t \in \mathrm{Symp}(M,\omega)$ is a symplectic diffeomorphism with compact support, and $\phi_0 = \mathrm{id}$. We denote by $\mathrm{Symp}_0(M,\omega)$ the set of all time-1 maps of compactly supported symplectic isotopies.

Isotopies $\Phi = \{\phi_t\}$ are in one-to-one correspondence with families of smooth vector fields $\{\dot{\phi}_t\}$ defined by

$$\dot{\phi}_t(x) = \frac{d\phi_t}{dt}(\phi_t^{-1}(x)).$$

If $\Phi \in \text{Iso}(M, \omega)$, then the one-form $i(\dot{\phi}_t)\omega$ such that

$$i(\dot{\phi}_t)\omega(X) = \omega(\dot{\phi}_t, X)$$

for all vector fields X is closed. If for all t the 1-form $i(\dot{\phi}_t)\omega$ is exact, that is, there exists a smooth function $F : M \times [0, 1] \to \mathbb{R}$, $F(x, t) = F_t(x)$, with compact supports such that $i(\dot{\phi}_t)\omega = dF_t$, then the isotopy Φ is called a Hamiltonian isotopy and will be denoted by Φ_F. We define the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms as the set of time-one maps of Hamiltonian isotopies.

For each $\Phi = \{\phi_t\} \in \text{Iso}(M, \omega)$, the mapping

$$\Phi \mapsto \left[\int_0^1 (i(\dot{\phi}_t)\omega) dt \right],$$

where $[\alpha]$ denotes the cohomology class of a closed form α, induces a well defined map $\tilde{S}$ from the universal cover of $\text{Symp}_0(M, \omega)$ to the first de Rham cohomology group $H^1(M, \mathbb{R})$. This map is called the **Calabi invariant** (or the **flux**). It is a surjective group homomorphism. Let $\Gamma \subset H^1(M, \mathbb{R})$ be the image by $\tilde{S}$ of the fundamental group of $\text{Symp}_0(M, \omega)$. We then get a surjective homomorphism

$$S : \text{Symp}_0(M, \omega) \to H^1(M, \mathbb{R})/\Gamma.$$

The kernel of this homomorphism is the group $\text{Ham}(M, \omega)$ [2, 3].

2.2. **The Hofer norm.** Hofer [14] defined the length l_H of a Hamiltonian isotopy Φ_F as

$$l_H(\Phi_F) = \int_0^1 (\text{osc}\, F_t(x))\, dt,$$

where the oscillation of a function $f : M \to \mathbb{R}$ is

$$\text{osc}\,(f) = \max_{x \in M}(f(x)) - \min_{x \in M}(f(x)).$$

For $\phi \in \text{Ham}(M, \omega)$, the **Hofer norm** of ϕ is

$$\|\phi\|_H = \inf\{l_H(\Phi_F)\},$$

where the infimum is taken over all Hamiltonian isotopies Φ_F with time-one map equal to ϕ, i.e. $\phi_{F,1} = \phi$.

The Hofer distance $d_H(\phi, \psi)$ between two Hamiltonian diffeomorphisms ϕ and ψ is

$$d_H(\phi, \psi) = \|\phi \circ \psi^{-1}\|_H.$$

This distance is bi-invariant. This property was used in [8] to prove Theorem 1.

2.3. **The Hofer-like norm.** Now let (M, ω) be a compact symplectic manifold without boundary, on which we fix a Riemannian metric g. For each $\Phi = \{\phi_t\} \in \text{Iso}(M, \omega)$, we consider the Hodge decomposition [19] of the 1-form $i(\dot{\phi}_t)\omega$ as

$$i(\dot{\phi}_t)\omega = \mathcal{H}_t + du_t,$$

where $\mathcal{H}_t$ is a harmonic 1-form. The forms $\mathcal{H}_t$ and u_t are unique and depend smoothly on t.

For $\Phi \in \text{Iso}(M, \omega)$, define

$$l_0(\Phi) = \int_0^1 (|\mathcal{H}_t| + \text{osc}\,(u_t(x))\; dt,$$

where $|\mathcal{H}_t|$ is a norm on the finite dimensional vector space of harmonic 1-forms. We let

$$l(\phi) = \frac{1}{2}(l_0(\Phi) + l_0(\Phi^{-1})),$$

where $\Phi^{-1} = \{\phi_t^{-1}\}$.

For each $\phi \in \text{Symp}_0(M, \omega)$, let

$$\|\phi\|_{HL} = \inf\{l(\Phi)\},$$

where the infimum is taken over all symplectic isotopies $\Phi = \{\phi_t\}$ with $\phi_1 = \phi$.

The following result was proved in [5].

Theorem 4. *For any closed symplectic manifold* (M, ω)*,* $\|\cdot\|_{HL}$ *is a norm on* $\text{Symp}_0(M, \omega)$*.*

Remark 5. The norm $\|\cdot\|_{HL}$ depends on the choice of the Riemannian metric g on M and the choice of the norm $|\cdot|$ on the space of harmonic 1-forms. However, different choices for g and $|\cdot|$ yield equivalent metrics. See Section 3 of [5] for more details.

2.4. **Some equivalence properties.** Let (M, ω) be a compact symplectic manifold. Buss and Leclercq have proved:

Theorem 6. [7] *The restriction of the Hofer-like norm* $\| \cdot \|_{HL}$ *to* $\mathrm{Ham}(M, \omega)$ *is equivalent to the Hofer norm* $\| \cdot \|_H$.

We now prove the following.

Theorem 7. *Let* $\phi \in \mathrm{Symp}_0(M, \omega)$. *The norm*

$$h \mapsto \|\phi \circ h \circ \phi^{-1}\|_{HL}$$

on $\mathrm{Symp}_0(M, \omega)$ *is equivalent to the norm* $\| \cdot \|_{HL}$.

Remark 8. We owe the statement of the above theorem to the referee of a previous version of this paper.

Proof. Let $\{h_t\}$ be an isotopy in $\mathrm{Symp}_0(M, \omega)$ from h to the identity, and let

$$i(\dot{h}_t)\omega = \mathcal{H}_t + du_t$$

be the Hodge decomposition of $i(\dot{h}_t)\omega$. Then $\Psi = \{\phi \circ h_t \circ \phi^{-1}\}$ is an isotopy from $\phi \circ h \circ \phi^{-1}$ to the identity and $\dot{\Psi}_t = \phi_* \dot{h}_t$. Therefore,

$$i(\dot{\Psi}_t)\omega = (\phi^{-1})^*(i(\dot{h}_t)\phi^*\omega) = (\phi^{-1})^*(\mathcal{H}_t + du_t) = (\phi^{-1})^*\mathcal{H}_t + d(u_t \circ \phi^{-1}).$$

Let $\{\phi_s^{-1}\}$ be an isotopy from ϕ^{-1} to the identity, and let $L_X = i_X d + d i_X$ be the Lie derivative in the direction X. Then

$$\frac{d}{ds}((\phi_s^{-1})^*\mathcal{H}_t) = (\phi_s^{-1})^*(L_{\dot{\phi}_s^{-1}}\mathcal{H}_t) = d((\phi_s^{-1})^* i(\dot{\phi}_s^{-1})\mathcal{H}_t),$$

where $\dot{\phi}_t^{-1} = (\frac{d}{dt}\phi_t^{-1}) \circ \phi_t$. Integrating from 0 to 1 we get

$$(\phi^{-1})^*\mathcal{H}_t - \mathcal{H}_t = d\alpha_t$$

where

$$\alpha_t = \int_0^1 ((\phi_s^{-1})^* i(\dot{\phi}_s^{-1})\mathcal{H}_t)\ ds.$$

Therefore,

$$i(\dot{\Psi}_t)\omega = \mathcal{H}_t + d(u_t \circ \phi^{-1} + \alpha_t).$$

Hence,

$$\begin{aligned} l_0(\Psi) &= \int_0^1 \big(|\mathcal{H}_t| + \mathrm{osc}\,(u_t \circ \phi^{-1} + \alpha_t)\big)\,dt \\ &\leq \int_0^1 \big(|\mathcal{H}_t| + \mathrm{osc}\,(u_t \circ \phi^{-1})\big)\,dt + \int_0^1 \mathrm{osc}\,(\alpha_t)\ dt \\ &= \int_0^1 (|\mathcal{H}_t| + \mathrm{osc}\,(u_t))\,dt + \int_0^1 \mathrm{osc}\,(\alpha_t)\ dt \\ &= l_0(\{h_t\}) + K \end{aligned}$$

where

$$K = \int_0^1 \mathrm{osc}\,(\alpha_t)\ dt.$$

Let us now do the same calculation for $\Psi^{-1} = \{\phi \circ h_t^{-1} \circ \phi^{-1}\}$.

Since $\dot{h}_t^{-1}$ satisfies $\dot{h}_t^{-1} = -(h_t^{-1})_*\dot{h}_t$, the cohomology classes of $i(\dot{h}_t)\omega$ and $i(\dot{h}_t^{-1})\omega$ are of opposite sign. Since the Hodge decomposition is unique and the harmonic part of the first form is $\mathcal{H}_t$, the harmonic part of the second form is $-\mathcal{H}_t$. Therefore, there is a smooth family of functions v_t such that the Hodge decomposition for $i(\dot{h}_t^{-1})\omega$ is

$$i(\dot{h}_t^{-1})\omega = -\mathcal{H}_t + dv_t.$$

The same calculation shows

$$i(\dot{\Psi}_t^{-1})\omega = -\mathcal{H}_t + d(v_t \circ \phi^{-1} - \alpha_t).$$

Hence,

$$l_0(\Psi^{-1}) \leq l_0(\{h_t^{-1}\}) + K.$$

We will now estimate $K = \int_0^1 \mathrm{osc}\,(\alpha_t)\ dt$. Fix an isotopy $\{\phi_s^{-1}\}$ from ϕ^{-1} to the identity. Consider the continuous linear map

$$\mathcal{L}_{\{\phi_s^{-1}\}} : \mathcal{H}^1(M, g) \to C^\infty(M)$$

from the finite dimensional vector space of harmonic 1-forms given by

$$\mathcal{L}_{\{\phi_s^{-1}\}}(\theta) = \int_0^1 ((\phi_s^{-1})^* i(\dot{\phi}_s^{-1})\theta)\ ds.$$

Let $\nu \geq 0$ be the norm of $\mathcal{L}_{\{\phi_s^{-1}\}}$ where the norm on $\mathcal{H}^1(M,g)$ is defined by the metric g and $C^\infty(M)$ is given the sup norm. Then $|\mathcal{L}_{\{\phi_s^{-1}\}}(\theta)| \leq \nu|\theta|$. In our case $\alpha_t = \mathcal{L}_{\{\phi_s^{-1}\}}(\mathcal{H}_t)$. Therefore,

$$|\alpha_t| \leq \nu|\mathcal{H}_t|$$

and

$$\mathrm{osc}\,(\alpha_t) \leq 2|\alpha_t| \leq 2\nu|\mathcal{H}_t|.$$

This implies

$$\mathrm{osc}\,(\alpha_t) \leq 2\nu\left(|\mathcal{H}_t| + \mathrm{osc}\,(u_t)\right) \text{ and } \mathrm{osc}\,(\alpha_t) \leq 2\nu\left(|\mathcal{H}_t| + \mathrm{osc}\,(v_t)\right).$$

Hence,

$$K = \int_0^1 \mathrm{osc}\,(\alpha_t)\; dt \leq 2\nu\, l_0(\{h_t\}),$$

and

$$K = \int_0^1 \mathrm{osc}\,(\alpha_t)\; dt \leq 2\nu\, l_0(\{h_t^{-1}\}).$$

Now recall that,

$$l_0(\Psi) \leq l_0(\{h_t\}) + K \text{ and } l_0(\Psi^{-1}) \leq l_0(\{h_t^{-1}\}) + K.$$

Therefore,

$$\begin{aligned} l(\Psi) &= \frac{1}{2}\left(l_0(\Psi) + l_0(\Psi^{-1})\right) \\ &\leq \frac{1}{2}\left(l_0(\{h_t\}) + 2\nu\, l_0(\{h_t\}) + l_0(\{h_t^{-1}\}) + 2\nu\, l_0(\{h_t^{-1}\})\right) \\ &\leq (2\nu+1)l(\{h_t\}). \end{aligned}$$

Taking the infimum over the set $I(h)$ of all symplectic isotopies from h to the identity we get

$$\inf_{I(h)} l(\Psi) \leq (2\nu+1)\|h\|_{HL},$$

and since

$$\|\phi \circ h \circ \phi^{-1}\|_{HL} \leq \inf_{I(h)} l(\Psi)$$

we get

$$\|\phi \circ h \circ \phi^{-1}\|_{HL} \leq k\|h\|_{HL}$$

with $k = 2\nu + 1$.

We have shown that for every $\phi \in \mathrm{Symp}_0(M, \omega)$ there is a $k \geq 1$ (depending on an isotopy $\{\phi_s\}$ from ϕ to the identity) such that the preceding inequality holds for all $h \in \mathrm{Symp}_0(M, \omega)$. Applying this to ϕ^{-1} we see that there is an $k' \geq 1$ such that

$$\|\phi^{-1} \circ h \circ \phi\|_{HL} \leq k' \|h\|_{HL}$$

for all $h \in \mathrm{Symp}_0(M, \omega)$. Therefore, for any $h \in \mathrm{Symp}_0(M, \omega)$ we have

$$\|h\|_{HL} = \|\phi^{-1} \circ (\phi \circ h \circ \phi^{-1}) \circ \phi\|_{HL} \leq k' \|\phi \circ h \circ \phi^{-1}\|_{HL}.$$

That is,

$$\frac{1}{k'} \|h\|_{HL} \leq \|\phi \circ h \circ \phi^{-1}\|_{HL} \leq k \|h\|_{HL}.$$

□

Remark 9. The constant k depends only on ϕ^{-1} rather than the isotopy $\{\phi_s^{-1}\}$, because the function $\mathcal{L}_{\{\phi_s^{-1}\}}(\theta)$ is the unique normalized function on M such that $d(\mathcal{L}_{\{\phi_s^{-1}\}}(\theta)) = (\phi^{-1})^* \theta - \theta$.

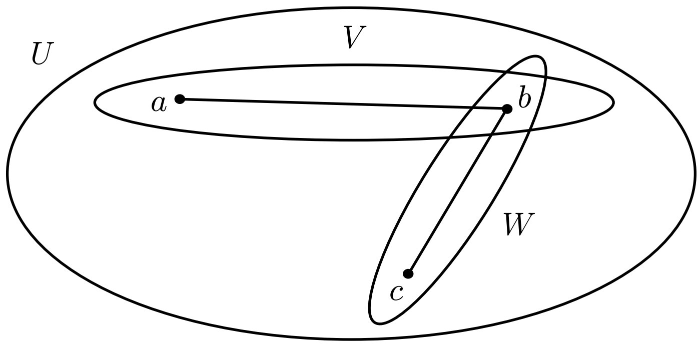
3. Proof of the main result

We will closely follow the proof given by Polterovich of Theorem 2.4.A in [17] that $e(A) > 0$. We will use without any change Proposition 1.5.B.

Proposition 1.5.B. [17] *For any non-empty open subset A of M, there exists a pair of Hamiltonian diffeomorphisms ϕ and ψ that are supported in A and whose commutator $[\phi, \psi] = \psi^{-1} \circ \phi^{-1} \circ \psi \circ \phi$ is not equal to the identity.*

For the sake of completeness we provide the following alternate proof of this proposition based on the transitivity lemmas in [3] (pages 29 and 109). (For a proof of k-fold transitivity for symplectomorphisms see [6].)

Proof. Let U be an open connected subset of A such that $\overline{U} \subset A$. Pick three distinct points $a, b, c \in U$. By the transitivity lemma of $\mathrm{Ham}(M, \omega)$, there exist $\phi, \psi \in \mathrm{Ham}(M, \omega)$ such that $\phi(a) = b$ and $\psi(b) = c$. Moreover, we can choose ϕ and ψ so that $\mathrm{supp}\,(\phi)$ and $\mathrm{supp}\,(\psi)$ are contained in small tubular neighborhoods V and W of distinct paths in U joining a to b and b to c respectively, and we can assume that $c \in U \backslash V$.



Then $(\psi^{-1}\phi^{-1}\psi\phi)(a) = (\psi^{-1}\phi^{-1})(c) = \psi^{-1}(c) = b$. Hence $[\phi, \psi] \neq \mathrm{id}$. □

We will say that a map h **displaces** A if $h(A) \cap A = \emptyset$. Let us denote by $D(A)$ the set of all $h \in \mathrm{Symp}_0(M, \omega)$ that displace A. We note the following fact.

Lemma 10. *Let ϕ and ψ be as in Proposition 1.5.B, and let $h \in D(A)$. Then the commutator*

$$\theta = [h, \phi^{-1}] = \phi \circ h^{-1} \circ \phi^{-1} \circ h$$

satisfies $[\phi, \psi] = [\theta, \psi]$.

Proof. If $x \in A$ then $h(x) \notin A$. Hence,

$$\begin{aligned} \theta(x) &= (\phi \circ h^{-1})(\phi^{-1}(h(x))) \\ &= \phi(h^{-1}(h(x))) \qquad \text{since supp}\,(\phi^{-1}) \subset A \\ &= \phi(x), \end{aligned}$$

and we see that $\theta|_A = \phi|_A$. Similarly, for $x \in A$ we have $\phi^{-1}(x) \in A$, and hence $h(\phi^{-1}(x)) \notin A$ since $h(A) \cap A = \emptyset$. Thus,

$$\begin{aligned} \theta^{-1}(x) &= h^{-1}(\phi(h(\phi^{-1}(x)))) \\ &= h^{-1}(h(\phi^{-1}(x))) \qquad \text{since supp}\,(\phi) \subset A \\ &= \phi^{-1}(x), \end{aligned}$$

and we see that $\theta^{-1}|_A = \phi^{-1}|_A$. Thus, $(\phi^{-1} \circ \psi \circ \phi)(x) = (\theta^{-1} \circ \psi \circ \theta)(x)$ for all $x \in A$ since $\mathrm{supp}\,(\psi) \subset A$.

Now, if $x \notin A$ and $\theta(x) \in A$ we would have $x = \theta^{-1}(\theta(x)) = \phi^{-1}(\theta(x)) \in A$ since supp $(\phi^{-1}) \subset A$, a contradiction. Hence, for $x \notin A$ we have $\theta(x) \notin A$ and

$$(\phi^{-1} \circ \psi \circ \phi)(x) = x = (\theta^{-1} \circ \psi \circ \theta)(x)$$

since both ϕ and ψ have support in A. Therefore, $\phi^{-1} \circ \psi \circ \phi = \theta^{-1} \circ \psi \circ \theta$, and we have $[\phi, \psi] = [\theta, \psi]$.

□

Proof of Theorem 3 continued. Following the proof of Theorem 2.4.A in [17] we assume there exists $h \in D(A) \neq \emptyset$. Otherwise, we are done since $e_s(A) = +\infty$. Now, let ϕ and ψ be as in Proposition 1.5.B, and let θ be as in Lemma 10. The commutator θ is contained in $\mathrm{Ham}(M, \omega)$ because commutators are in the kernel of the Calabi invariant. Since both θ and ψ are in $\mathrm{Ham}(M, \omega)$ and the Hofer norm is conjugation invariant, we have

$$\begin{aligned} \|[\theta, \psi]\|_H &= \|\psi^{-1} \circ \theta^{-1} \circ \psi \circ \theta\|_H \\ &\leq \|\psi^{-1} \circ \theta^{-1} \circ \psi\|_H + \|\theta\|_H \\ &= 2\|\theta\|_H. \end{aligned}$$

By Buss and Leclercq's theorem [7] there is constant $\lambda > 0$ such that

$$\|\theta\|_H \leq \lambda \|\theta\|_{HL}.$$

Using the triangle inequality and the constant $k > 0$ from Theorem 7 we have

$$\begin{aligned} \|[\theta, \psi]\|_H &\leq 2\lambda \left(\|\phi \circ h \circ \phi^{-1}\|_{HL} + \|h\|_{HL} \right) \\ &\leq 2\lambda \left(k\|h\|_{HL} + \|h\|_{HL} \right). \end{aligned}$$

Therefore,

$$0 < \frac{\|[\phi, \psi]\|_H}{2\lambda(k+1)} = \frac{\|[\theta, \psi]\|_H}{2\lambda(k+1)} \leq \|h\|_{HL}.$$

Since this inequality holds for all $h \in D(A)$, we can take the infimum over $D(A)$ to get

$$0 < \frac{\|[\phi, \psi]\|_H}{2\lambda(k+1)} \leq e_s(A).$$

This completes the proof of Theorem 3. □

Remark 11. The proof of Theorem 1 relied on the bi-invariance of the distance d_H, whereas the proof of Theorem 3 relied on the equivalence of the norms $h \mapsto \|\phi \circ h \circ \phi^{-1}\|_{HL}$ and $\|\cdot\|_{HL}$, i.e. the invariance of d_{HL} up to a constant.

4. Examples

A harmonic 1-parameter group is an isotopy $\Phi = \{\phi_t\}$ generated by the vector field $V_{\mathcal{H}}$ defined by $i(V_{\mathcal{H}})\omega = \mathcal{H}$, where $\mathcal{H}$ is a harmonic 1-form. It is immediate from the definitions that

$$l_0(\Phi) = l_0(\Phi^{-1}) = |\mathcal{H}|$$

where $|\cdot|$ is a norm on the space of harmonic 1-forms. Hence $l(\Phi) = |\mathcal{H}|$. Therefore, if ϕ_1 is the time one map of Φ we have

$$\|\phi_1\|_{HL} \leq |\mathcal{H}|.$$

For instance, take the torus T^{2n} with coordinates $(\theta_1, \dots, \theta_{2n})$ and the flat Riemannian metric. Then all the 1-forms $d\theta_i$ are harmonic. Given $v = (a_1, \dots, a_n, b_1, \dots, b_n) \in \mathbb{R}^{2n}$, the translation $x \mapsto x + v$ on $\mathbb{R}^{2n}$ induces a rotation ρ_v on T^{2n}, which is a symplectic diffeomorphism. Moreover, $x \mapsto x + tv$ on $\mathbb{R}^{2n}$ induces a harmonic 1-parameter group $\{\rho_v^t\}$ on T^{2n}.

Taking the 1-forms $d\theta_i$ for $i = 1, \dots, 2n$ as basis for the space of harmonic 1-forms and using the standard symplectic form

$$\omega = \sum_{j=1}^{n} d\theta_j \wedge d\theta_{j+n}$$

on T^{2n} we have

$$i(\dot{\rho_v^t})\omega = \sum_{j=1}^{n} \left(a_j d\theta_{j+n} - b_j d\theta_j\right).$$

Thus,

$$l(\{\rho_v^t\}) = |(-b_1, \dots, -b_n, a_1, \dots, a_n)|$$

where $|\cdot|$ is a norm on the space of harmonic 1-forms, and we see that

$$\|\rho_v\|_{HL} \leq |v|$$

if we use $|v| = |a_1| + \cdots + |a_n| + |b_1| + \cdots + |b_n|$ as the norm on both $\mathbb{R}^{2n}$ and the space of harmonic 1-forms.

Consider the torus T^2 as the square:

$$\{(p,q) \mid 0 \leq p \leq 1 \text{ and } 0 \leq q \leq 1\} \subset \mathbb{R}^2$$

with opposite sides identified. For any $r < \frac{1}{2}$ let

$$\tilde{A}(r) = \{(x,y) \mid 0 \leq x < r\} \subset \mathbb{R}^2,$$

and let $A(r)$ be the corresponding subset in T^2. If $v = (r, 0)$, then the rotation ρ_v induced by the translation $(p,q) \mapsto (p+r, q)$ displaces $A(r)$. Therefore, using the norm $|v| = |a_1| + |b_1| = r$ we have

$$\|\rho_v\|_{HL} \leq l(\{\rho_v^t\}) = r.$$

Therefore,

$$e_s(A(r)) \leq r.$$

Remark 12. Note that in the above example the symplectic displacement energy is finite, whereas the Hamiltonian displacement energy $e(A(r))$ is infinite. This follows from a result proved by Gromov [12]: If (M, ω) is a symplectic manifold without boundary that is convex at infinity and $L \subset M$ is a compact Lagrangian submanifold such that $[\omega]$ vanishes on $\pi_2(M, L)$, then for any Hamiltonian symplectomorphism $\phi : M \to M$ the intersection $\phi(L) \cap L \neq \emptyset$. Stronger versions of this result can be found in [9], [10], and [11]. See also Section 9.2 of [15].

5. Application

The following result is an immediate consequence of the positivity of the symplectic displacement energy of non-empty open sets. For two isotopies Φ and Ψ denote by $\Phi^{-1} \circ \Psi$ the isotopy given at time t by $(\Phi^{-1} \circ \Psi)_t = \phi_t^{-1} \circ \psi_t$.

Theorem 13. *Let Φ_n be a sequence of symplectic isotopies and let Ψ be another symplectic isotopy. Suppose that the sequence of time-one maps $\phi_{n,1}$ of the isotopies Φ_n converges uniformly to a homeomorphism ϕ, and $l(\Phi_n^{-1} \circ \Psi) \to 0$ as $n \to \infty$, then $\phi = \psi_1$.*

This theorem can be viewed as a justification for the following definition, which appeared in [1] and [4].

Definition 14. A homeomorphism h of a compact symplectic manifold is called a **strong symplectic homeomorphism** if there exist a sequence Φ_n of symplectic isotopies such that $\phi_{n,1}$ converges uniformly to h, and $l(\Phi_n)$ is a Cauchy sequence.

Proof of Theorem 13. Suppose $\phi \neq \psi_1$, i.e. $\phi^{-1} \circ \psi_1 \neq \mathrm{id}$. Then there exists a small open ball B such that $(\phi^{-1} \circ \psi_1)(\overline{B}) \cap \overline{B} = \emptyset$. Since $\phi_{n,1}$ converges uniformly to ϕ, $((\phi_{n,1})^{-1} \circ \psi_1)(B) \cap B = \emptyset$ for n large enough. Therefore, the symplectic displacement energy $e_s(B)$ of B satisfies

$$e_s(B) \leq \|(\phi_{n,1})^{-1} \circ \psi_1\|_{HL} \leq l(\Phi_n^{-1} \circ \Psi).$$

The last term tends to zero, which contradicts the positivity of $e_s(B)$. □

Remark 15. This theorem was first proved by Hofer and Zehnder for $M = \mathbb{R}^{2n}$ [13], and then by Oh-Müller in [16] for Hamiltonian isotopies using the same lines as above, and very recently by Tchuiaga [18], using the L^∞ version of the Hofer-like norm.

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