COMPACTIFIED SPACES OF HOLOMORPHIC CURVES IN COMPLEX GRASSMANN MANIFOLDS

DAVID E. HURTUBISE AND MARC D. SANDERS

ABSTRACT. In this paper we study the topology of a compactification of the space of holomorphic maps of fixed degree from $\mathbb{C}P^1$ into a finite dimensional complex Grassmann manifold. We show that there is a homotopy equivalence through a range, increasing with the degree, between these compact spaces and an infinite dimensional complex Grassmann manifold. These compact spaces form a direct system indexed by the degree, and the direct limit is homotopy equivalent to an infinite dimensional complex Grassmann manifold.

1. INTRODUCTION

In [15] Segal proved that the inclusion of the space of all holomorphic maps from $\mathbb{C}P^1$ to $\mathbb{C}P^k$ of degree $d \in \mathbb{N}$ into the space of all continuous maps from S^2 to $\mathbb{C}P^k$ of degree d is a homotopy equivalence through a range that increases with d. He proved this result for both based and unbased maps. Segal also proved a similar result in homology for spaces of unbased maps from a general Riemann surface into $\mathbb{C}P^k$. In [10] Kirwan extended Segal's result for homology to spaces of unbased maps from a Riemann surface into a finite dimensional complex Grassmann manifold, and in [2] Boyer, J.C. Hurtubise, Mann, and Milgram extended Segal's result in both homology and homotopy to maps from $\mathbb{C}P^1$ into an arbitrary flag manifold.

Segal's original work was motivated by the role that spaces of holomorphic maps play in control theory. A complex controllable and observable linear system is a system of differential equations of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

where A, B, and C are matrices with entries in $\mathbb{C}$ [7]. Two controllable and observable linear systems (A, B, C) and (A', B', C') are said to be

1991 *Mathematics Subject Classification.* Primary: 55P10.
Key words and phrases. holomorphic curves, compactification, classifying space.

state space equivalent if there exists a matrix S such that

$$(A, B, C) \sim (SAS^{-1}, SB, CS^{-1}) = (A', B', C').$$

The space of based holomorphic maps from $\mathbb{C}P^1$ to $G_{n,n+k}(\mathbb{C})$ of degree d is in one-to-one correspondence with the set of state space equivalence classes of controllable and observable linear systems of McMillan degree d [13]. The homology of the space of based holomorphic maps from $\mathbb{C}P^1$ to $G_{n,n+k}(\mathbb{C})$ of degree d gives information about the complexity of the moduli space of controllable and observable linear systems of McMillan degree d, and it is this connection that motivated Mann and Milgram to compute the homology of the space of holomorphic maps of degree d from $\mathbb{C}P^1$ to $G_{n,n+k}(\mathbb{C})$ [12].

These mapping spaces also have compactifications which are of considerable interest in control theory, as well as symplectic geometry and gauge theory. In [3] Byrnes constructed a compactification of the space of based holomorphic maps of degree d from $\mathbb{C}P^1$ to $G_{n,n+k}(\mathbb{C})$ by embedding this mapping space into a large finite dimensional complex Grassmann manifold $G_{n,(d+1)(n+k)}(\mathbb{C})$ and then taking the closure of the mapping space inside $G_{n,(d+1)(n+k)}(\mathbb{C})$. Byrnes used this compactification to obtain new results on pole placement by degree one compensators, which are still considered "state of the art" [8]. In [11] Kirwan computed the Betti numbers of a compactification of the moduli space of stable vector bundles of fixed rank and fixed degree over a compact Riemann surface of genus $g \geq 2$. More recently, Bertram, Daskalopoulos, and Wentworth constructed three different compactifications of the space of holomorphic maps of fixed degree from a Riemann surface of genus g into a finite dimensional complex Grassmann manifold [1]. They then used these compactifications to provide a framework for calculating Donaldson-type Gromov invariants.

In this paper we study the topology of a compactification of the space of *un*based holomorphic maps of degree d from $\mathbb{C}P^1$ to $G_{n,n+k}(\mathbb{C})$. This compactification is the analogue of Byrnes' compactification. We prove that this compactification $\overline{\mathrm{Hol}}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is homotopy equivalent to the classifying space $BGL_n(\mathbb{C})$ through a range that increases with d. More specifically, we prove the following theorem.

Theorem 1. *There exists a map* $\overline{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to BGL_n(\mathbb{C})$ *which induces isomorphisms in homotopy groups through dimension* $\binom{(d+1)(n+k)}{n} - \binom{(d+1)(n-1)}{n} - 2$.

Moreover, the compactified spaces form a direct system with respect to the degree and in the limit we obtain a homotopy equivalence.

Corollary 2. *The direct limit of*

$$\overline{Hol_1}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \overline{Hol_2}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \cdots$$

is homotopy equivalent to $BGL_n(\mathbb{C})$.

2. The Compactification

Let $M_{n,n+k}(\mathbb{C}[z])$ be the set of $n \times (n+k)$ matrices with entries in the polynomial ring $\mathbb{C}[z]$. The group $GL_n(\mathbb{C}[z])$, consisting of all $n \times n$ matrices with polynomial entries whose determinant is a nonzero constant, acts on $M_{n,n+k}(\mathbb{C}[z])$ by multiplication on the left.

$$GL_n(\mathbb{C}[z]) \times M_{n,n+k}(\mathbb{C}[z]) \to M_{n,n+k}(\mathbb{C}[z])$$

Let $P_{n,n+k}(\mathbb{C}[z])$ denote the space of polynomial maps from $\mathbb{C}$ to the Stiefel manifold $V_{n,n+k}(\mathbb{C})$. A matrix in $M_{n,n+k}(\mathbb{C}[z])$ has $\binom{n+k}{n}$ minors of size $n \times n$. The determinants of these minors are in $\mathbb{C}[z]$. $P_{n,n+k}(\mathbb{C}[z])$ is the subspace of $M_{n,n+k}(\mathbb{C}[z])$ consisting of those matrices such that these $\binom{n+k}{n}$ polynomials do not all have a root in common. Since multiplying an element of $M_{n,n+k}(\mathbb{C}[z])$ on the left by an element of $GL_n(\mathbb{C}[z])$ can only change the determinants of the $n \times n$ minors by an element of $\mathbb{C}\backslash\{0\}$, the above action restricts to an action on $P_{n,n+k}(\mathbb{C}[z])$.

$$GL_n(\mathbb{C}[z]) \times P_{n,n+k}(\mathbb{C}[z]) \to P_{n,n+k}(\mathbb{C}[z])$$

Moreover, this action restricts to the subspace $P^d_{n,n+k}(\mathbb{C}[z])$ consisting of those matrices whose $n \times n$ determinants are all of degree less than or equal to degree d with at least one determinant having degree d.

The following result is well known [9] [12].

Theorem 3. *The space of holomorphic maps of degree d from $\mathbb{C}P^1$ to the complex Grassmann manifold $G_{n,n+k}(\mathbb{C})$ with the compact open topology is homeomorphic to $P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ topologized with the quotient topology.*

$$Hol_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \approx P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$$

Note that when $n = 1$ this theorem says that there exists an embedding of the mapping space $\mathrm{Hol}_d(\mathbb{C}P^1, \mathbb{C}P^k)$ into $\mathbb{C}P^{(d+1)(1+k)-1}$ because $GL_1(\mathbb{C}[z]) = \mathbb{C}\backslash\{0\}$. When $n > 1$ a similar embedding exists of $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ into $G_{n,(d+1)(n+k)}(\mathbb{C})$. To define this embedding we will first describe a normal form for matrices in the quotient space $P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$.

The action

$$GL_n(\mathbb{C}[z]) \times M_{n,n+k}(\mathbb{C}[z]) \to M_{n,n+k}(\mathbb{C}[z])$$

corresponds to polynomial row operations on an element of $M_{n,n+k}(\mathbb{C}[z])$. That is, by multiplying an element of $M_{n,n+k}(\mathbb{C}[z])$ on the left by an element of $GL_n(\mathbb{C}[z])$ we can interchange rows, multiply a row by a non-zero constant, or add a polynomial multiple of one row to another row [5]. Let $(p_{ij}(z)) \in M_{n,n+k}(\mathbb{C}[z])$. By multiplying $(p_{ij}(z))$ on the left by elements of $GL_n(\mathbb{C}[z])$ we can put $(p_{ij}(z))$ into the following polynomial reduced row echelon form:

$$\begin{pmatrix} p_{1\,1} & \cdots & p_{1\,i_1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ p_{2\,1} & \cdots & p_{2\,i_1} & p_{2\,i_1+1} & \cdots & p_{2\,i_2} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ p_{n\,1} & \cdots & p_{n\,i_1} & p_{n\,i_1+1} & \cdots & p_{n\,i_2} & p_{n\,i_2+1} & \cdots & p_{n\,i_n} & 0 & \cdots & 0 \end{pmatrix}$$

where $i_1, \ldots, i_n$ is a strictly increasing sequence of integers between 1 and $n+k$, the rightmost polynomial in each row is a non-zero monic polynomial, and the polynomials below each of $p_{1\,i_1}, p_{2\,i_2}, \ldots, p_{n-1\,i_{n-1}}$ all have degree strictly less than the degree of $p_{1\,i_1}, p_{2\,i_2}, \ldots, p_{n-1\,i_{n-1}}$ respectively. By considering the possible polynomial row operations induced by the action of $GL_n(\mathbb{C}[z])$ on $M_{n,n+k}(\mathbb{C}[z])$ it is easy to see that each orbit contains a unique matrix in polynomial reduced row echelon form.

It is well known that $P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ is a complex analytic manifold with local coordinate charts given by the coefficients of the polynomials of a matrix in reduced row echelon form [4] [12]. The transition functions are given by multiplying by elements of $GL_n(\mathbb{C}[z])$ which can be solved for using Cramer's Rule. This complex analytic structure on $P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ is defined analogous to the complex analytic structure on $V_{n,(d+1)(n+k)}(\mathbb{C})/GL_n(\mathbb{C}) = G_{n,(d+1)(n+k)}(\mathbb{C})$ [6]. In both cases the topology defined by these local coordinate charts agrees with the quotient topology.

Let $d \in \mathbb{N}$ and let $\mathbb{C}^d[z]$ denote the vector space of polynomials of degree less than or equal to d. It is easy to show that a matrix $M \in P^d_{n,n+k}(\mathbb{C}[z])$ which is in polynomial reduced row echelon form has entries in $\mathbb{C}^d[z]$ and therefore has rows that can be thought of as elements of the complex vector space

$$\underbrace{\mathbb{C}^d[z] \times \cdots \times \mathbb{C}^d[z]}_{n+k} \approx \mathbb{C}^{(d+1)(n+k)}.$$

Since $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \approx P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ we can define a map

$$\phi : \mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to G_{n,(d+1)(n+k)}(\mathbb{C})$$

by sending a holomorphic curve σ to the plane in $\mathbb{C}^{(d+1)(n+k)}$ spanned by the rows of the unique polynomial reduced row echelon form matrix in the orbit corresponding to σ. The above observations concerning the complex analytic structures on $P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ and $G_{n,(d+1)(n+k)}(\mathbb{C})$ make the proof of the following theorem immediate.

Theorem 4. *ϕ is a complex analytic embedding.*

To compactify $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ we take the closure of this space embedded in $G_{n,(d+1)(n+k)}(\mathbb{C})$.

Definition 5.

$$\overline{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) = \overline{\phi(Hol_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})))} \subseteq G_{n,(d+1)(n+k)}(\mathbb{C})$$

Note that points in $G_{n,(d+1)(n+k)}(\mathbb{C})$ can be represented by elements in the vector space $M_{n,n+k}(\mathbb{C}[z])$ (modulo the action of $GL_n(\mathbb{C})$) and so it makes sense to talk about the $n \times n$ determinants computed by treating each polynomial as a single entry in the matrix. We will refer to these $\binom{n+k}{n}$ determinants as the polynomial determinants of an element of $G_{n,(d+1)(n+k)}(\mathbb{C})$. They are well defined up to an element of $\mathbb{C}\backslash\{0\}$.

Theorem 6. *$\overline{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is the set of points in the closure of the Schubert cell $e((d+1)(1+k), \dots, (d+1)(n+k))$ in the complex Grassmann manifold $G_{n,(d+1)(n+k)}(\mathbb{C})$ whose $n \times n$ polynomial determinants are all of degree less than or equal to d. When $n = 1$ we have the following.*

$$\overline{Hol}_d(\mathbb{C}P^1, \mathbb{C}P^k) = \mathbb{C}P^{(d+1)(1+k)-1}$$

Proof:

It is clear that a matrix in $P^d_{n,n+k}(\mathbb{C}[z])$ which is in polynomial reduced row echelon form has rows that span a plane in the closure of the Schubert cell $e((d+1)(1+k), \dots, (d+1)(n+k))$ (Our notation for Schubert cells follows [14] Section 6.) Therefore,

$$\phi(\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \subseteq \overline{e((d+1)(1+k), \dots, (d+1)(n+k))}.$$

Since the $n \times n$ polynomial determinants of an element of $M_{n,n+k}(\mathbb{C}[z])$ can only have finitely many zeros in common, $P_{n,n+k}(\mathbb{C}[z])$ is dense in $M_{n,n+k}(\mathbb{C}[z])$. Hence $\phi(\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is dense in the set of points in the closure of $e((d+1)(1+k), \dots, (d+1)(n+k))$ whose $n \times n$ polynomial determinants are all of degree less than or equal to d.

For the case $n = 1$ simply note that the Schubert cell $e((d+1)(1+k))$ is dense in $\mathbb{C}P^{(d+1)(1+k)-1}$ and the condition on the $n \times n$ polynomial determinants is vacuous when $n = 1$.

□

3. The homotopy groups of $\overline{\mathrm{Hol}}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$

In this section we prove that $\overline{\mathrm{Hol}}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ and $BGL_n(\mathbb{C})$ are homotopy equivalent through a range that increases with d.

Pick any $d \in \mathbb{N}$ and recall that $\mathbb{C}^d[z]$ is the vector space of all complex polynomials of degree less than or equal to d and $M_{n,n+k}(\mathbb{C}^d[z])$ is the set of all $n \times (n+k)$ matrices with elements in $\mathbb{C}^d[z] \approx \mathbb{C}^{(d+1)}$. Let $M^d_{n,n+k}(\mathbb{C}^d[z])$ denote the subset of $M_{n,n+k}(\mathbb{C}^d[z])$ consisting of those matrices whose $n \times n$ polynomial determinants are all contained in $\mathbb{C}^d[z]$. The condition for an element $\alpha \in M_{n,n+k}(\mathbb{C}^d[z])$ to be in $M^d_{n,n+k}(\mathbb{C}^d[z])$ is an algebraic condition given by $\binom{n+k}{n}(dn-d)$ polynomial equations in the coefficients of the polynomial entries of α. Note that each $\alpha \in M^d_{n,n+k}(\mathbb{C}^d[z])$ can be written as

$$\alpha = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_{n+k})$$

where each α_i can be thought of as an element of $M_{n,d+1}(\mathbb{C})$. Let $C^d_{n,n+k}$ be the collection of those $\alpha \in M^d_{n,n+k}(\mathbb{C}^d[z])$ that satisfy the following condition: all $j \times j$ complex determinants of the matrix

$$(\alpha_{n+k+2-j} \ \alpha_{n+k+2-j+1} \ \cdots \ \alpha_{n+k}) \in M_{n,(d+1)(j-1)}(\mathbb{C})$$

are zero for all $2 \leq j \leq n$. Thus $C^d_{n,n+k}$ is a subset of $M^d_{n,n+k}(\mathbb{C}^d[z])$ determined by a finite set of polynomial equations in the coefficients of the polynomial entries. By considering the standard basis for a point in a Schubert cell (see [14] Section 6) and standard complex row reduction techniques it is not hard to see that the preimage of

$$\overline{\mathrm{Hol}}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \subseteq G_{n,(d+1)(n+k)}(\mathbb{C})$$

under the quotient map

$$\pi : V_{n,(d+1)(n+k)}(\mathbb{C}) \to G_{n,(d+1)(n+k)}(\mathbb{C}).$$

is $C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C})$.

Lemma 7. *$C^d_{n,n+k}$ is contractible.*

Proof:

Note that for $c \in \mathbb{C}$ and $\alpha \in C^d_{n,n+k}$ we have $c\alpha \in C^d_{n,n+k}$. So, for $t \in [0,1]$ we can define $F_t : C^d_{n,n+k} \to C^d_{n,n+k}$ by $F_t(\alpha) = t\alpha$. This gives the desired homotopy, and we see that the space contracts to the zero matrix.

□

Lemma 8.

$$H_j(C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C})) = 0$$

for all $j < \binom{(d+1)(n+k)}{n} - \binom{(d+1)(n-1)}{n} - 1$.

Proof:

Let D be the subset of $C^d_{n,n+k}$ consisting of those matrices whose rows are linearly dependent as elements of the complex vector space $\mathbb{C}^{(d+1)(n+k)}$.

$$C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C}) = C^d_{n,n+k} - D$$

D is the closed analytic subvariety of $C^d_{n,n+k}$ consisting of those matrices whose $n \times n$ complex minors are all zero. Since all the $n \times n$ minors of the rightmost $(d+1)(n-1)$ columns of a matrix in $C^d_{n,n+k}$ are already zero, D is the intersection of $C^d_{n,n+k}$ with the zero set of an additional $\binom{(d+1)(n+k)}{n} - \binom{(d+1)(n-1)}{n}$ polynomials. By a result of F. Kirwan, [10] Theorem 6.1, it follows that

$$H_j(C^d_{n,n+k} - D) = H_j(C^d_{n,n+k}) = 0$$

for all $j < \binom{(d+1)(n+k)}{n} - \binom{(d+1)(n-1)}{n} - 1$.

□

Lemma 9. $C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C})$ *is simply connected. Hence,*

$$\pi_j(C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C})) = 0$$

for all $j < \binom{(d+1)(n+k)}{n} - \binom{(d+1)(n-1)}{n} - 1$.

Proof:

Fix an ordering on the $\binom{(d+1)(n+k)}{n}$ complex $n \times n$ minors of the matrices in $C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C})$ and let $I_1, \dots, I_m$ be the indices in this ordering corresponding to those minors such that there exists a matrix in $C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C})$ whose I_jth minor has non-zero determinant. For every $1 \leq j \leq m$ let U_{I_j} be the collection of those matrices in $C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C})$ whose I_jth $n \times n$ minor has non-zero determinant, union an additional matrix M. M is chosen to be a matrix with all entries zero except for one $n \times n$ complex minor equal to the identity matrix in columns corresponding to the constant terms of the polynomial entries.

Any loop in U_{I_j} can be contracted to M. One way of contracting such a loop is to first perturb the loop so that it does not go through M, then contract all the entries outside the I_jth minor to zero, and then contract the entries in the I_jth minor to zero while continuously deforming the other $n \times n$ complex minor to the identity matrix. Therefore, $\pi_1(U_{I_j}) = 0$ for all $1 \leq j \leq m$.

It is easy to see that $U_{I_1} \cap U_{I_2}$ is path connected and hence by the Seifert-Van Kampen Theorem $\pi_1(U_{I_1} \cup U_{I_2}) = 0$. By considering

$$(U_{I_1} \cup U_{I_2}) \cap U_{I_3} = (U_{I_1} \cap U_{I_3}) \cup (U_{I_2} \cap U_{I_3})$$

one can easily see that $(U_{I_1} \cup U_{I_2}) \cap U_{I_3}$ is path connected and hence $\pi_1(U_{I_1} \cup U_{I_2} \cup U_{I_3}) = 0$. Continuing this process one sees that

$$\pi_1(U_{I_1} \cup \cdots \cup U_{I_m}) = \pi_1(C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C})) = 0.$$

The last statement follows from the Hurewicz Theorem and Lemma 8.

□

Theorem 10. *There exists a map $\overline{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to BGL_n(\mathbb{C})$ which induces isomorphisms in homotopy groups through dimension $\binom{(d+1)(n+k)}{n} - \binom{(d+1)(n-1)}{n} - 2$.*

Proof:

The principle bundle

$$\begin{array}{ccc} GL_n(\mathbb{C}) & \longrightarrow & C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C}) \\ & & \downarrow \pi \\ & & \overline{\mathrm{Hol}}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \end{array}$$

induces the following fibration.

$$\begin{array}{ccc} C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C}) & \xrightarrow{\pi} & \overline{\mathrm{Hol}}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \\ & & \downarrow \\ & & BGL_n(\mathbb{C}) \end{array}$$

By the previous lemma $\pi_j(C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C})) = 0$ for all $j < \binom{(d+1)(n+k)}{n} - \binom{(d+1)(n-1)}{n} - 1$ and hence the map

$$\overline{\mathrm{Hol}}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to BGL_n(\mathbb{C})$$

induces isomorphisms in homotopy groups through the same range.

□

4. The direct limit of the compactified spaces

The inclusion

$$M_{n,n+k}(\mathbb{C}^d[z]) \to M_{n,n+k}(\mathbb{C}^{d+1}[z])$$

induces an inclusion of Stiefel manifolds

$$V_{n,(d+1)(n+k)}(\mathbb{C}) \to V_{n,(d+2)(n+k)}(\mathbb{C})$$

and Grassmann manifolds

$$G_{n,(d+1)(n+k)}(\mathbb{C}) \to G_{n,(d+2)(n+k)}(\mathbb{C})$$

which in turn induces an inclusion

$$\overline{\mathrm{Hol}}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \overline{\mathrm{Hol}}_{d+1}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$$

by Theorem 6. These inclusion maps determine the following direct system.

$$\overline{\mathrm{Hol}_1}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \overline{\mathrm{Hol}_2}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \cdots$$

The results of the previous section show that the homotopy direct limit of this system is homotopy equivalent to $BGL_n(\mathbb{C})$. We now give a relatively quick self-contained proof that the ordinary direct limit is homotopy equivalent to $BGL_n(\mathbb{C})$.

Let $\widetilde{\mathrm{Hol}}_d$ denote the preimage of

$$\overline{\mathrm{Hol}}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \subseteq G_{n,(d+1)(n+k)}(\mathbb{C})$$

under the quotient map

$$\pi : V_{n,(d+1)(n+k)}(\mathbb{C}) \to G_{n,(d+1)(n+k)}(\mathbb{C}).$$

In other words, $\widetilde{\mathrm{Hol}}_d$ is $C^d_{n,n+k} \cap V_{n,(d+1)(n+k)}(\mathbb{C})$ from the previous section.

Lemma 11. *The direct limit of*

$$\widetilde{Hol}_1 \to \widetilde{Hol}_2 \to \widetilde{Hol}_3 \to \cdots$$

is contractible.

Proof:

It suffices to show that the homotopy groups of the direct limit are all zero. Pick any $d \in \mathbb{N}$. The inclusion

$$i_{d,1} : \widetilde{\mathrm{Hol}}_d \to \widetilde{\mathrm{Hol}}_{d+1}$$

can be described as follows. An element $\alpha \in \widetilde{\mathrm{Hol}}_d \subseteq V_{(d+1)(n+k)}(\mathbb{C})$ is an $n \times (d+1)(n+k)$ matrix. For each $1 \le i \le n+k$ let α_i denote the ith $n \times (d+1)$ block.

$$\alpha = (\alpha_1\ \alpha_2\ \cdots\ \alpha_{n+k})$$

$i_{d,1}(\alpha)$ is the matrix obtained by placing a column of zeroes in front of each α_i.

$$i_{d,1}(\alpha) = (0_{n\times 1}\alpha_1\ \ 0_{n\times 1}\alpha_2\ \ \cdots\ \ 0_{n\times 1}\alpha_{n+k})$$

The inclusion

$$i_{d,n^2+nd} : \widetilde{\mathrm{Hol}}_d \to \widetilde{\mathrm{Hol}}_{d+n^2+nd}$$

is the composition $i_{d,n^2+nd} = i_{d+n^2+nd-1,1} \circ \cdots \circ i_{d,1}$. Thus,

$$i_{d,n^2+nd}(\alpha) = (0_{n\times(n^2+nd)}\alpha_1 \quad 0_{n\times(n^2+nd)}\alpha_2 \quad \cdots \quad 0_{n\times(n^2+nd)}\alpha_{n+k})$$

where $0_{n\times(n^2+nd)}$ is the $n \times (n^2+nd)$ zero matrix. The map

$$H_{d,n^2+nd} : \widetilde{\mathrm{Hol}}_d \times [0,1] \to \widetilde{\mathrm{Hol}}_{d+n^2+nd}$$

defined by $H_{d,n^2+nd}(\alpha, t) =$

$$(0_{n\times n^2+nd-n}(1-t)I_{n\times n}t\alpha_1 \quad 0_{n\times(n^2+nd)}t\alpha_2 \quad \cdots \quad 0_{n\times(n^2+nd)}t\alpha_{n+k})$$

where $I_{n\times n}$ is the $n \times n$ identity matrix is a homotopy from i_{d,n^2+nd} to a constant map. Therefore the homotopy groups of the direct limit are all zero.

□

Theorem 12.

$$\varinjlim_{d} \overline{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \simeq BGL_n(\mathbb{C})$$

Proof:

The group action of $GL_n(\mathbb{C})$ on $\widetilde{\mathrm{Hol}}_d$ commutes with the inclusion maps. Therefore the quotient of the direct limit of

$$\widetilde{\mathrm{Hol}}_1 \to \widetilde{\mathrm{Hol}}_2 \to \cdots$$

is the direct limit of the quotient spaces

$$\overline{\mathrm{Hol}_1}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \overline{\mathrm{Hol}_2}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \cdots$$

Since $GL_n(\mathbb{C})$ acts freely on the contractible space $\varinjlim_{d} \widetilde{\mathrm{Hol}}_d$, the quotient space of this direct limit is homotopy equivalent to $BGL_n(\mathbb{C})$.

□

References

[1] A. Bertram, G. Daskalopoulos, and R. Wentworth. Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians. *J. Amer. Math. Soc.*, 9(2):529–571, 1996.

[2] C.P. Boyer, J.C. Hurtubise, B.M. Mann, and R.J. Milgram. The topology of the space of rational maps into generalized flag manifolds. *Acta Math.*, 173:61–101, 1994.

[3] C.I. Byrnes. On compactifications of spaces of systems and dynamic compensation. In *Proceedings of the 22nd IEEE Conference on Decision and Control*, pages 889–894, 1983.

[4] J.M.C. Clark. The consistent selection of parameterizations in system identification. In *Proc. Joint Automatic Control Conference*, pages 576–580, 1976.

[5] F.R. Gantmacher. *The Theory of Matricies Vol. 1.* Chelsea Publishing Company, New York, 1960.

[6] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* John Wiley and Sons, New York, 1978.

[7] U. Helmke. *The cohomology of moduli spaces of linear dynamical systems.* Regensburger Mathematische Schriften, Habilitationsschrift Regensburg University, 1992.

[8] U. Helmke. A compactification of the space of rational transfer functions by singular systems. *J. Math. Systems Estim. Control*, 3(4):459–472, 1993.

[9] D.E. Hurtubise. Spaces of holomorphic maps from $\mathbb{C}P^1$ to complex Grassmann manifolds. In A. Banyaga et. al., editor, *Topics in Low Dimensional Topology*, pages 101–113. World Scientific, 1999.

[10] F. Kirwan. On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles. *Ark. Mat.*, 24(2):221–275, 1986.

[11] F. Kirwan. On the homology of compactifications of moduli spaces of vector bundles over a Riemann surface. *Proc. London Math. Soc.*, 53(3):237–266, 1986.

[12] B. Mann and R.J. Milgram. Some spaces of holomorphic maps to complex Grassmann manifolds. *J. Differential Geom.*, 33(2):301–324, 1991.

[13] C. Martin and R. Hermann. Applications of algebraic geometry to systems theory: The McMillan degree and Kronecker indices of transfer functions as topological and holomorphic system invariants. *SIAM J. Control Optim.*, 16(5):743–755, 1978.

[14] J.W. Milnor and J.D. Stasheff. *Characteristic Classes.* Princeton University Press, Princeton, New Jersey, 1974.

[15] G. Segal. The topology of spaces of rational functions. *Acta Math.*, 143:39–72, 1979.

Department of Mathematics, Penn State University – The Altoona College, 101B Eiche, Altoona, PA 16601-3760

E-mail address: hurtubis@math.psu.edu

Department of Mathematics, Stanford University, Stanford, CA 94305

E-mail address: sanders@csli.stanford.edu