AN ALGEBRAIC COMPACTIFICATION FOR SPACES OF HOLOMORPHIC CURVES IN COMPLEX GRASSMANN MANIFOLDS

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Abstract. We construct a compactification of the space of holomorphic curves of fixed degree in a finite dimensional complex Grassmann manifold using basic algebra. The algebraic compactification is defined as the quotient of n-tuples of linearly independent elements in a $\mathbb{C}[z]$-module. The complex analytic structure on the space of holomorphic curves of fixed degree extends to the algebraic compactification. We show that there is a homotopy equivalence through a range increasing with the degree between the compactified spaces and an infinite dimensional complex Grassmann manifold. These compact spaces form a direct system, indexed by the degree, whose direct limit is homotopy equivalent to an infinite dimensional complex Grassmann manifold.

1. Introduction

Let Σ be a compact Riemann surface, and let X be a complex analytic manifold. Let $\mathrm{Hol}_d(\Sigma, X)$ denote the space of holomorphic maps of degree d from Σ to X, and let $\mathrm{Hol}_d^0(\Sigma, X)$ denote the space of based holomorphic maps of degree d. There is a long list of results concerning the topology of the mapping spaces $\mathrm{Hol}_d(\Sigma, X)$ and $\mathrm{Hol}_d^0(\Sigma, X)$. Most of these results have to do with proving homology and/or homotopy equivalences between these spaces and the space of all continuous maps of degree d (based or unbased) through a range that increases with d. Results have been proved for $\mathrm{Hol}_d(\mathbb{C}P^1, \mathbb{C}P^k)$ [27], $\mathrm{Hol}_d(\Sigma, G_{n,n+k}(\mathbb{C}))$ [20], $\mathrm{Hol}_d^0(\mathbb{C}P^1, G/P)$ [4], and $\mathrm{Hol}_d^0(\Sigma, G/P)$ [18], where G/P is a generalized flag manifold. Similar stability results have been proved for spaces of instantons over certain four manifolds [3] [19].

In many cases one can define a gluing operation such that the spaces $\mathrm{Hol}_d(\Sigma, X)$ and $\mathrm{Hol}_d^0(\Sigma, X)$ form a direct system with respect to the degree d. In [7] Cohen, Jones, and Segal began a general investigation of what they call the *stability property* for a space X. They say that

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a space X has the stability property if there is an appropriate limiting process so that

$$\lim_{\vec{d}} \mathrm{Hol}_d^0(\mathbb{C}P^1, X) \simeq \Omega^2 X.$$

They conjecture that if X is a closed, simply connected, integral symplectic manifold, then the stability property holds if and only if the evaluation map $E : \lim_d \mathrm{Hol}_d^0(\mathbb{C}P^1, X) \to X$ is a quasifibration.

The space $\mathrm{Hol}_d^0(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ and certain open dense subsets of $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ are of interest in linear control theory because they are homeomorphic to spaces of equivalence classes of controllable and observable linear systems of McMillan degree d. A complex *controllable and observable linear system* is a system of differential equations of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

where A, B, C, and D are matrices with entries in $\mathbb{C}$. Two controllable and observable linear systems (A, B, C, D) and (A', B', C', D') are said to be *state space equivalent* if $D = D'$ and there exists a matrix S such that

$$(A, B, C) \sim (SAS^{-1}, SB, CS^{-1}) = (A', B', C').$$

The space of state space equivalence classes of controllable and observable linear systems of McMillan degree d with $D = 0$ is homeomorphic to $\mathrm{Hol}_d^0(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ [12]. The full space of state space equivalence classes of controllable and observable linear systems of McMillan degree d is homeomorphic to the open dense subset of $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ consisting of those maps which send the base point of $\mathbb{C}P^1$ into the largest dimensional Schubert cell of $G_{n,n+k}(\mathbb{C})$. The homology of the mapping spaces gives information about the complexity of the moduli spaces of controllable and observable linear systems. It is this connection with linear control theory that motivated Mann and Milgram to compute the homology of $\mathrm{Hol}_d^0(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ [21].

In general, the space $\mathrm{Hol}_d(\Sigma, X)$ is not compact. However, the space still has a fundamental homology class which can be used to define Gromov-Witten invariants. One approach to defining the Gromov-Witten invariants rigorously involves compactifying mapping spaces such as $\mathrm{Hol}_d(\Sigma, X)$ in such a way that the boundary component has co-dimension 2. Bertram, Daskalopoulos, and Wentworth constructed three different compactifications of $\mathrm{Hol}_d(\Sigma, G_{n,n+k}(\mathbb{C}))$ in order to provide a framework for calculating Donaldson-type Gromov invariants [2], and Ruan and Tian used the Gromov compactification for moduli spaces of pseudo-holomorphic curves in order to give a rigorous definition of the Gromov-Witten invariants for all semi-positive symplectic manifolds [26].

Compactifications of the mapping spaces $\mathrm{Hol}_d^0(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ and $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ are of interest in linear control theory to people who study the identification problem and the pole placement problem. In [5] Byrnes constructed a compactification of the space of proper transfer functions (which is homeomorphic to the open dense subset of $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ described above) and used his compactification to obtain new results on pole placement by degree one compensators which are still considered "state of the art" [13].

In [17] we began to study the stability properties of compactifications of $\mathrm{Hol}_d(\Sigma, X)$ and $\mathrm{Hol}_d^0(\Sigma, X)$. We studied the topology of Byrnes' compactification for $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ and proved that the Byrnes compactification is homotopy equivalent to the classifying space $BGL_n(\mathbb{C})$ through a range that increases with d. More specifically, we proved the following theorem.

Theorem 1. *There exists a map* $\overline{Hol}_d^B(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to BGL_n(\mathbb{C})$ *which induces isomorphisms in homotopy groups through dimension* $\binom{(d+1)(n+k)}{n} - \binom{(d+1)(n-1)}{n} - 2$.

The Byrnes compactification may be important in linear systems theory with regard to the pole placement problem, but it isn't the most natural compactification from our point of view. In this paper we construct a different compactification of the space $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ which follows naturally from the description of the space given in [16] and some basic algebra. In addition to being a much more natural compactification (from our point of view) than the Byrnes compactification, the algebraic compactification has the advantage of being a complex analytic manifold. $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is a complex analytic manifold with local coordinates given by the free coefficients of matrices that are in canonical form [6] [21]. In Section 3 we show that these local charts naturally extend to the algebraic compactification.

The algebraic compactification, $\overline{\mathrm{Hol}}_d^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$, satisfies a stability property similar to that of the Byrnes compactification. In Section 5 we prove the following theorem.

Theorem 2. *There exists a map* $\overline{Hol}_d^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to BGL_n(\mathbb{C})$ *which induces isomorphisms in homotopy groups through dimension* $\binom{n+k}{n}(d+1) - 2$.

As with the Byrnes compactification, the algebraic compactification gives a direct system with respect to the degree and in the limit we obtain a homotopy equivalence.

Corollary 3. *The direct limit of*

$$\overline{Hol}_1^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \overline{Hol}_2^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \cdots$$

is homotopy equivalent to $BGL_n(\mathbb{C})$.

2. The Algebraic Compactification

Let $M_{n,n+k}(\mathbb{C}[z])$ be the set of $n \times (n+k)$ matrices with entries in the polynomial ring $\mathbb{C}[z]$. The group $GL_n(\mathbb{C}[z])$, consisting of all $n \times n$ matrices with polynomial entries whose determinant is a non-zero constant, acts on $M_{n,n+k}(\mathbb{C}[z])$ by multiplication on the left.

$$GL_n(\mathbb{C}[z]) \times M_{n,n+k}(\mathbb{C}[z]) \to M_{n,n+k}(\mathbb{C}[z])$$

Let $P_{n,n+k}(\mathbb{C}[z])$ denote the space of polynomial maps from $\mathbb{C}$ to the Stiefel manifold $V_{n,n+k}(\mathbb{C})$. A matrix in $M_{n,n+k}(\mathbb{C}[z])$ has $\binom{n+k}{n}$ minors of size $n \times n$. The determinants of these minors are in $\mathbb{C}[z]$. $P_{n,n+k}(\mathbb{C}[z])$ is the subspace of $M_{n,n+k}(\mathbb{C}[z])$ consisting of those matrices such that these $\binom{n+k}{n}$ polynomials do not all have a root in common. Since multiplying an element of $M_{n,n+k}(\mathbb{C}[z])$ on the left by an element of $GL_n(\mathbb{C}[z])$ can only change the determinants of the $n \times n$ minors by an element of $\mathbb{C} - \{0\}$, the above action restricts to an action

$$GL_n(\mathbb{C}[z]) \times P_{n,n+k}(\mathbb{C}[z]) \to P_{n,n+k}(\mathbb{C}[z]).$$

Moreover, this action restricts to the subspace $P^d_{n,n+k}(\mathbb{C}[z])$ consisting of those matrices whose $n \times n$ determinants are all of degree less than or equal to d with at least one determinant having degree d.

The following result is well known [16] [21].

Theorem 4. *The space of holomorphic maps of degree d from $\mathbb{C}P^1$ to the complex Grassmann manifold $G_{n,n+k}(\mathbb{C})$ with the compact open topology is homeomorphic to $P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ topologized with the quotient topology.*

$$Hol_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \approx P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$$

To form the algebraic compactification we relax the condition that says the rows of a matrix in $P^d_{n,n+k}(\mathbb{C}[z])$ must be linearly independent at every point $z \in \mathbb{C}$. Let $V_{n,n+k}(\mathbb{C}[z])$ denote the set of all $n \times (n+k)$

matrices with entries in the ring $\mathbb{C}[z]$ whose rows are linearly independent elements of the $\mathbb{C}[z]-$module

$$\underbrace{\mathbb{C}[z] \times \cdots \times \mathbb{C}[z]}_{n+k}.$$

Let $V^d_{n,n+k}(\mathbb{C}[z])$ be the subspace consisting of those matrices whose $n \times n$ determinants are all of degree less than or equal to d.

Definition 5. *The algebraic compactification of $Hol_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is defined to be*

$$\overline{Hol}^A_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) = V^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z]).$$

Note that a matrix in $P^d_{n,n+k}(\mathbb{C}[z])$ must have at least one $n \times n$ minor whose determinant is of degree d, whereas a matrix in $V^d_{n,n+k}(\mathbb{C}[z])$ may have all of its $n \times n$ determinants of degree strictly less than d. Thus $\overline{\mathrm{Hol}}^A_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ contains all holomorphic curves of degree less than or equal to d.

Claim 6. *The quotient space $V^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ is well-defined and contains $P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ as an open and dense subset.*

To establish Claim 6, we'll first prove a lemma that will be used again in Section 3. Let $M^d_{n,n+k}(\mathbb{C}[z])$ denote the set of $n \times (n+k)$ matrices with entries in $\mathbb{C}[z]$ whose $n \times n$ determinants are all of degree less than or equal to d.

Lemma 7. *Each of the following inclusions is open and dense*

$$P^d_{n,n+k}(\mathbb{C}[z]) \subset V^d_{n,n+k}(\mathbb{C}[z]) \subset M^d_{n,n+k}(\mathbb{C}[z]).$$

Proof:

Let $M \in M^d_{n,n+k}(\mathbb{C}[z])$. M is in $V^d_{n,n+k}(\mathbb{C}[z])$ if and only if at least one of its $n \times n$ determinants is not zero. This condition is clearly open and dense.

An element $M \in M_{n,n+k}(\mathbb{C}[z])$ is in $P_{n,n+k}(\mathbb{C}[z])$ if and only if the $n \times n$ determinants do not all have a root in common. Since these $\binom{n+k}{n}$ polynomials can only have finitely many roots in common, it is possible to perturb the entries of M slightly so that these polynomials do not all have a root in common. Moreover, if the $n \times n$ determinants of M do not have any roots in common, then neither will the $n \times n$ determinants of a slight perturbation of M. Thus the inclusion

$$P_{n,n+k}(\mathbb{C}[z]) \subset M_{n,n+k}(\mathbb{C}[z])$$

is open and dense, and the same holds when we include the restrictions on the degrees of the $n \times n$ determinants.

□

Proof of Claim 6:

The action

$$GL_n(\mathbb{C}[z]) \times M_{n,n+k}(\mathbb{C}[z]) \to M_{n,n+k}(\mathbb{C}[z])$$

corresponds to polynomial row operations. That is, by multiplying an element of $M_{n,n+k}(\mathbb{C}[z])$ on the left by an element of $GL_n(\mathbb{C}[z])$ one can interchange rows, multiply a row by a non-zero constant, or add a polynomial multiple of one row to another row [8]. Hence this action restricts to an action on $V_{n,n+k}(\mathbb{C}[z])$ and also to

$$GL_n(\mathbb{C}[z]) \times V^d_{n,n+k}(\mathbb{C}[z]) \to V^d_{n,n+k}(\mathbb{C}[z])$$

since multiplying an element of $M_{n,n+k}(\mathbb{C}[z])$ on the left by an element of $GL_n(\mathbb{C}[z])$ can only change the determinants of the $n \times n$ minors by an element of $\mathbb{C} - \{0\}$. This shows that the quotient space is well defined.

By Lemma 7, $P^d_{n,n+k}(\mathbb{C}[z])$ is open and dense in $V^d_{n,n+k}(\mathbb{C}[z])$, and hence the quotient space $P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ is open and dense in the quotient space $V^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$.

□

Note that the preceeding claim shows that $\text{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is open and dense in $\overline{\text{Hol}}^A_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$. In Section 4, where we discuss bubbling for pseudo-holomorphic curves, we will prove that the space $\overline{\text{Hol}}^A_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is compact, and therefore a compactification of $\text{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$.

3. Complex Analytic Coordinate Charts

It is well known that

$$\text{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \approx P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$$

is a complex analytic manifold with local coordinates given by the coefficients of the polynomial entries of a matrix in *canonical form* [6] [21]. The transition functions are given by multiplying by elements of $GL_n(\mathbb{C}[z])$ which can be solved for using Cramer's Rule. The complex analytic structure on $P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ is defined analogous to the complex analytic structure on $V_{n,n+k}(\mathbb{C})/GL_n(\mathbb{C}) = G_{n,n+k}(\mathbb{C})$ [10]. In both cases the topology defined by these local coordinate charts agrees with the quotient topology.

The following definition is taken from [14].

Definition 8. *Let G be a topological group which acts continuously on a space X. A canonical form for the group action*

$$G \times X \to X$$

is a map $\Gamma : X \to X$ such that for all $x, y \in X$

$$\begin{aligned} &(1) \qquad \Gamma(x) &= gx \qquad &\textit{for some } g \in G \\ &(2) \qquad x &= gy \qquad &\textit{for some } g \in G \textit{ iff } \ \Gamma(x) = \Gamma(y). \end{aligned}$$

Thus a canonical form for the group action

$$GL_n(\mathbb{C}[z]) \times P^d_{n,n+k}(\mathbb{C}[z]) \to P^d_{n,n+k}(\mathbb{C}[z])$$

is a map $\Gamma : P^d_{n,n+k}(\mathbb{C}[z]) \to P^d_{n,n+k}(\mathbb{C}[z])$ such that $\Gamma(M) = GM$ for some $G \in GL_n(\mathbb{C}[z])$ and $\Gamma(M) = \Gamma(N)$ if and only if M and N are in the same orbit. One such canonical form was defined by Mann and Milgram for based holomorphic maps in [21]. There is an analogous canonical form for unbased maps defined as follows.

The action

$$GL_n(\mathbb{C}[z]) \times M^d_{n,n+k}(\mathbb{C}[z]) \to M^d_{n,n+k}(\mathbb{C}[z])$$

corresponds to polynomial row operations. That is, by multiplying an element of $M^d_{n,n+k}(\mathbb{C}[z])$ on the left by an element of $GL_n(\mathbb{C}[z])$ we can interchange rows, multiply a row by a non-zero constant, or add a polynomial multiple of one row to another row [8]. Let $(p_{ij}(z))$ be an element of $M^d_{n,n+k}(\mathbb{C}[z])$. By multiplying $(p_{ij}(z))$ on the left by elements of $GL_n(\mathbb{C}[z])$ we can put $(p_{ij}(z))$ into the following polynomial reduced row echelon form

$$\begin{pmatrix} 0 & \cdots & 0 & p_{1\,j_1} & \cdots & p_{1\,j_2-1} & p_{1\,j_2} & \cdots & p_{1\,j_n-1} & p_{1\,j_n} & \cdots & p_{1\,n+k} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & p_{2\,j_2} & \cdots & p_{2\,j_n-1} & p_{2\,j_n} & \cdots & p_{2\,n+k} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & p_{n\,j_n} & \cdots & p_{n\,n+k} \end{pmatrix}$$

where $j_1, \ldots, j_n$ is a strictly increasing sequence of integers between 1 and $n + k$, the leftmost polynomial in each row is a non-zero monic polynomial, and the polynomials above $p_{2\,j_2}, p_{3\,j_3}, \ldots, p_{n\,j_n}$ all have degree strictly less than the degree of $p_{2\,j_2}, p_{3\,j_3}, \ldots, p_{n\,j_n}$ respectively. The above polynomial reduced row echelon form is also referred to as the *Hermite form* [1]. By considering the possible polynomial row operations induced by the action of $GL_n(\mathbb{C}[z])$ on $M^d_{n,n+k}(\mathbb{C}[z])$ it is easy to see that each orbit contains a unique matrix in Hermite form. Thus the Hermite form is a canonical form for the group action

$$GL_n(\mathbb{C}[z]) \times M^d_{n,n+k}(\mathbb{C}[z]) \to M^d_{n,n+k}(\mathbb{C}[z]).$$

Since this action restricts to an action on

$$GL_n(\mathbb{C}[z]) \times V^d_{n,n+k}(\mathbb{C}[z]) \to V^d_{n,n+k}(\mathbb{C}[z])$$

and also to an action on

$$GL_n(\mathbb{C}[z]) \times P^d_{n,n+k}(\mathbb{C}[z]) \to P^d_{n,n+k}(\mathbb{C}[z]),$$

the Hermite form is a canonical form for both of the restricted group actions as well.

By Lemma 7 the following inclusion is open and dense

$$P^d_{n,n+k}(\mathbb{C}[z]) \subset V^d_{n,n+k}(\mathbb{C}[z]),$$

and the above discussion shows that the Hermite form on $M^d_{n,n+k}(\mathbb{C}[z])$ restricts to a canonical form on both of these subspaces. Since local coordinates on $P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ are determined by the free coefficients of the polynomial entries of a matrix in canonical form [6] [21], we have the following result.

Theorem 9.

$$\overline{Hol}^A_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \approx V^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$$

is a complex analytic manifold. Local coordinates are determined by the free coefficients of the polynomials entries of a matrix in canonical form. The local coordinates restrict to local coordinates on

$$Hol_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \approx P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z]).$$

4. Bubbling for Pseudo-holomorphic Curves

Let (M,ω) be a symplectic manifold, and let J be an almost complex structure on M compatible with ω. Let $A \in H_2(M,\mathbb{Z})$ and let

$$\mathcal{M}(A,J) = \{f : \mathbb{C}P^1 \to M | f_*([\mathbb{C}P^1]) = A \text{ and } J \circ df = df \circ i\}$$

where i denotes the standard complex structure on $\mathbb{C}P^1$. A map f that satisfies $J \circ df = df \circ i$ is called a *pseudo-holomorphic curve* (or a *J-holomorphic curve*), and the space $\mathcal{M}(A,J)$ is called a *moduli space* of pseudo-holomorphic curves [23].

The compactness properties of $\mathcal{M}(A,J)$ have been studied by several authors. See for instance [11], [15], [24], [25], [28], and [29]. The space $\mathcal{M}(A,J)$ is usually not compact. However, non-compactness is always due to the appearance of "bubbles". To make this more precise consider a sequence $\{f_j\} \subset \mathcal{M}(A,J)$. The sequence $\{f_j\}$ will not necessarily have a convergent subsequence, but there will always be a subsequence of $\{f_j\}$ (still denoted by $\{f_j\}$) that converges in C^1 to a J-holomorphic curve $f : \mathbb{C}P^1 \to M$ off of finitely many points $p_1, \ldots, p_s \in \mathbb{C}P^1$. The points $p_1, \ldots, p_s$ are called *bubble points* because around each bubble point p a subsequence of $\{f_j\}$ can be rescaled to produce a *bubble map* $f_p : \mathbb{C}P^1 \to M$. The bubble map f_p is J-holomorphic, and it satisfies $f_p(\infty) = f(p)$. The rescaling process can be iterated to produce bubbles on bubbles. This leads to the notion of a *bubble tree* which is a finite collection of J-holomorphic curves defined on finitely many copies of $\mathbb{C}P^1$ connected at the bubble points. Bubble trees are used to define the *bubble tree compactification* of $\mathcal{M}(A, J)$, and several other useful compactifications of $\mathcal{M}(A, J)$ can be defined as quotients of the bubble tree compactification [24].

If we take $M = G_{n,n+k}(\mathbb{C})$ and let J be the standard complex structure on $G_{n,n+k}(\mathbb{C})$, then

$$\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) = \mathcal{M}(d[1], J)$$

where $[1] \in H_2(G_{n,n+k}(\mathbb{C}), \mathbb{Z}) \approx \mathbb{Z}$ is the positive generator. Thus we can apply the results of [24], [25], [28], and [29] to $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$. In particular, we have the following theorem as a special case of [25] Theorem 4.1.

Theorem 10. *Let $\{f_j\}$ be a sequence in $Hol_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$. There are finitely many points $p_1, \ldots, p_s \in \mathbb{C}P^1$ and a holomorphic map $f \in Hol_{\tilde{d}}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ for some $\tilde{d} \leq d$ such that, after passing to a subsequence, $\{f_j\}$ converges to f in C^1 on $\mathbb{C}P^1 - \{p_1, \ldots, p_s\}$.*

Theorem 11. *$\overline{Hol}_d^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is compact.*

Proof:

Since $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is dense in $\overline{\mathrm{Hol}}_d^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ it suffices to show that every sequence in $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}P^1))$ has a subsequence that converges to an element of $\overline{\mathrm{Hol}}_d^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$. Let $\{f_j\}$ be a sequence in $\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$. By Theorem 10 there exist finitely many points $p_1, \ldots, p_s$ such that a subsequence of $\{f_j\}$ converges in C^1 to some holomorphic map $f \in \mathrm{Hol}_{\tilde{d}}(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ on $\mathbb{C}P^1 - \{p_1, \ldots, p_s\}$. Let $\{f_j\}$ denote this subsequence.

By Theorem 4 we have

$$\mathrm{Hol}_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \approx P^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z]).$$

Under this homeomorphism a matrix $M_j \in P^d_{n,n+k}(\mathbb{C}[z])$ represents a holomorphic map $f_j : \mathbb{C}P^1 \to G_{n,n+k}(\mathbb{C})$ if and only if for every $z \in \mathbb{C}P^1$ the rows of $M_j(z) \in V_{n,n+k}(\mathbb{C})$ span the plane $f_j(z)$. Recall that $G_{n,n+k}(\mathbb{C}) \approx V_{n,n+k}(\mathbb{C})/GL_n(\mathbb{C})$ has $\binom{n+k}{n}$ local coordinate charts indexed by the $n \times n$ minors [10]. The image of f intersects at least one of the open sets U_I defining the local coordinate charts. To simplify the notation we will assume that $I = (1, \ldots, n)$. For $J \in \mathbb{N}$ large enough, there is an open set $U \subseteq \mathbb{C}P^1 - \{p_1, \ldots, p_s\}$ such that for all $z \in U$, $f(z) \in U_I$ and $f_j(z) \in U_I$ for all $j > J$. In the local chart on U_I we have for all $z \in U$

$$\begin{pmatrix} 1 & & 0 & \\ & \ddots & & T_j(z) \\ 0 & & 1 & \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & & 0 & \\ & \ddots & & T(z) \\ 0 & & 1 & \end{pmatrix} \text{ as } j \to \infty$$

where $T_j(z)$ and $T(z)$ are $n \times k$ matrices of rational functions. For all $z \in U$ the rows of the matrix on the left span the plane $f_j(z)$, and the rows of the matrix on the right span the plane $f(z)$.

We can now follow the construction of Hermann and Martin to replace the above representations of $f_j(z)$ and $f(z)$ with matrices of polynomials rather than matrices of rational functions [22]. Choose a matrix fractional decomposition $T(z) = D^{-1}(z)N(z)$ with $D(z)$ and $N(z)$ left coprime [1]. Since $T_j(z) \to D^{-1}(z)N(z)$ as $j \to \infty$, there exist left coprime matrix fractional decompositions $T_j(z) = D_j^{-1}(z)N_j(z)$ and an $n \times n$ matrix of polynomials $G(z)$ such that $D_j(z) \to G(z)D(z)$ and $N_j(z) \to G(z)N(z)$ as $j \to \infty$. Hence there exist matrices $M_j = (D_j \ N_j) \in P^d_{n,n+k}(\mathbb{C}[z])$, such that $[M_j] = f_j$, a matrix $M = (D \ N) \in P^{\tilde{d}}_{n,n+k}(\mathbb{C}[z])$, such that $[M] = f$, and an $n \times n$ matrix of polynomials G such that

$$\lim_{j \to \infty} M_j = GM \in M^d_{n,n+k}(\mathbb{C}[z]).$$

The determinants of the $n \times n$ minors of M are polynomials that do not have any roots in common, and the determinants of the $n \times n$ minors of GM are these same polynomials multiplied by $\det G \in \mathbb{C}[z]$. If $\det G$ were identically zero, then the rows of $G(z)M(z)$ would be linearly dependent for every z. Since the rows of $G(z)M(z)$ span an n-plane for every $z \neq p_1, \ldots, p_s$, $\det G$ must be a non-zero polynomial, and the rows of GM are linearly independent as elements of the $\mathbb{C}[z]-$module $\underbrace{\mathbb{C}[z] \times \cdots \times \mathbb{C}[z]}_{n+k}$. Therefore

$$\lim_{j \to \infty} [M_j] = [GM] \in V^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$$

and the space $\overline{\mathrm{Hol}}^A_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ is compact.

□

The proof of the preceeding theorem gives some insight into how the algebraic compactification, $\overline{\mathrm{Hol}}^A_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$, is related to the bubble tree compactification of Parker and Wolfson [24] [25] and the

Gromov compactification [28] [29]. When $n = 1$ the algebraic compactification is easily seen to be a quotient of the bubble tree compactification. The algebraic compactification keeps track of the layer one bubble points and the degree of the bubble tower above each layer one bubble point $p_1, \dots, p_s \in \mathbb{C}P^1$. More explicitly, suppose that

$$f_j(z) = (p_0^j(z) : \cdots : p_k^j(z)) \in \mathrm{Hol}_d(\mathbb{C}P^1, \mathbb{C}P^k)$$

converges to $(p_0(z) : \cdots : p_k(z)) \in V^d_{1,1+k}(\mathbb{C}[z])/GL_1(\mathbb{C}[z])$. If the limit of $\{f_j\}$ is not a holomorphic map, then the polynomials $p_0(z), \dots, p_k(z)$ have some common roots $p_1, \dots, p_s \in \mathbb{C}P^1$. These common roots are the layer one bubble points, and the degree of the common root is the total degree of the bubble tower above the bubble point.

When $n > 1$ the algebraic compactification keeps track of the layer one bubble points and the total degree of the bubble tower above each layer one bubble point. However, not all bubble maps with bubble towers of degree $d_1, \dots, d_s$ above the bubble points $p_1, \dots, p_s$ are identified. Some information about the image of the bubble maps is retained in $\overline{\mathrm{Hol}}^A_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$.

To see this, consider an element $[M] \in V^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$ and let M be the unique representative of $[M]$ in reduced row echelon form. If $[M]$ does not represent a holomorphic map, then some of the rows of M have polynomial entries that share common roots. The common roots $p_1, \dots, p_s$ and the degrees of these roots correspond to the bubble points and the degree of the bubble towers in the bubble tree compactification, as in the case $n = 1$. However, if we move some of the common factors from one row of M to another, then we get a different element of $V^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z])$. The information concerning which rows of M contain the common factors can be related to the image of the bubble maps, but this relationship is rather complicated in general.

5. The homotopy groups of $\overline{\mathrm{HOL}}^A_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$

In this section we prove that $\overline{\mathrm{Hol}}^A_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C}))$ and $BGL_n(\mathbb{C})$ are homotopy equivalent through a range that increases with d.

Let $M^d_{n,n+k}(\mathbb{C}[z])$ be the set of all $n \times (n+k)$ matrices with elements in $\mathbb{C}[z]$ whose $n \times n$ determinants are all of degree less than or equal to d.

Lemma 12. *$M^d_{n,n+k}(\mathbb{C}[z])$ is contractible.*

Proof:

Define $\phi : M^d_{n,n+k}(\mathbb{C}[z]) \times [0,1] \to M^d_{n,n+k}(\mathbb{C}[z])$ by

$$\phi((p_{ij}(z)),t) = (p_{ij}(tz)).$$

When $t = 0$ the image lies in $M_{n,n+k}(\mathbb{C}) \approx \mathbb{C}^{n(n+k)}$.

□

Lemma 13.

$$H_j(V^d_{n,n+k}(\mathbb{C}[z])) = 0$$

for all $j < \binom{n+k}{n}(d+1) - 1$.

Proof:

Let D be the subset of $M^d_{n,n+k}(\mathbb{C}[z])$ consisting of those matrices whose rows are linearly dependent elements of the $\mathbb{C}[z]-$module

$$\underbrace{\mathbb{C}[z] \times \cdots \times \mathbb{C}[z]}_{n+k}.$$

Note that $V^d_{n,n+k}(\mathbb{C}[z]) = M^d_{n,n+k}(\mathbb{C}[z]) - D$.

D is the closed analytic subvariety of $M^d_{n,n+k}(\mathbb{C}[z])$ consisting of those matrices whose $n \times n$ determinants are all zero. Thus D is the intersection of $M^d_{n,n+k}(\mathbb{C}[z])$ with the zero set of $\binom{n+k}{n}(d+1)$ polynomials in the coefficients of the $n \times n$ determinants. By a result of F. Kirwan, [20] Theorem 6.1, it follows that

$$H_j(M^d_{n,n+k}(\mathbb{C}[z]) - D) = H_j(M^d_{n,n+k}(\mathbb{C}[z])) = 0$$

for all $j < \binom{n+k}{n}(d+1) - 1$.

□

Lemma 14. $V^d_{n,n+k}(\mathbb{C}[z])$ *is simply connected. Hence,*

$$\pi_j(V^d_{n,n+k}(\mathbb{C}[z])) = 0$$

for all $j < \binom{n+k}{n}(d+1) - 1$.

Proof:

Let $\sigma : S^1 \to V^d_{n,n+k}(\mathbb{C}[z])$ be a continuous map. The subset of $V^d_{n,n+k}(\mathbb{C}[z])$ consisting of those matrices whose $n \times n$ determinants all have constant term zero is a subvariety of complex co-dimension $\binom{n+k}{n}$. Hence we can deform σ so that for every $t \in S^1$ the matrix $\sigma(t)$ has at least one $n \times n$ determinant with a non-zero constant term [9]. Define $\phi : V^d_{n,n+k}(\mathbb{C}[z]) \times [0,1] \to V^d_{n,n+k}(\mathbb{C}[z])$ by $\phi((p_{ij}(z)),t) = (p_{ij}(tz))$. $\phi \circ \sigma$ contracts σ into $V_{n,n+k}(\mathbb{C})$ which is simply connected.

The last statement follows from the Hurewicz Theorem and Lemma 13.

□

Lemma 15. *$GL_n(\mathbb{C})$ is a deformation retract of $GL_n(\mathbb{C}[z])$ and*
$$BGL_n(\mathbb{C}) \simeq BGL_n(\mathbb{C}[z]).$$

Proof:
Define $\phi : GL_n(\mathbb{C}[z]) \times [0,1] \to GL_n(\mathbb{C}[z])$ by $\phi((p_{ij}(z),t) = (p_{ij}(tz))$. When $t = 0$ the image lies in $GL_n(\mathbb{C})$. So $GL_n(\mathbb{C})$ is a deformation retract of $GL_n(\mathbb{C}[z])$. Thus the inclusion
$$GL_n(\mathbb{C}) \to GL_n(\mathbb{C}[z])$$
is a homotopy equivalence which induces a homotopy equivalence
$$BGL_n(\mathbb{C}) \to BGL_n(\mathbb{C}[z]).$$

□

Theorem 16. *There exists a map $\overline{Hol}_d^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to BGL_n(\mathbb{C})$ which induces isomorphisms in homotopy groups through dimension $\binom{n+k}{n}(d+1) - 2$.*

Proof:
The principal bundle

$$\begin{array}{ccc} GL_n(\mathbb{C}[z]) & \longrightarrow & V^d_{n,n+k}(\mathbb{C}[z]) \\ & & \downarrow \pi \\ & & \overline{\mathrm{Hol}}_d^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \end{array}$$

induces the following fibration.

$$\begin{array}{ccc} V^d_{n,n+k}(\mathbb{C}[z]) & \xrightarrow{\pi} & \overline{\mathrm{Hol}}_d^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \\ & & \downarrow \\ & & BGL_n(\mathbb{C}[z]) \end{array}$$

By Lemma 14, $\pi_j(V^d_{n,n+k}(\mathbb{C}[z])) = 0$ for all $j < \binom{n+k}{n}(d+1) - 1$ and hence the map
$$\overline{\mathrm{Hol}}_d^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to BGL_n(\mathbb{C}[z])$$
induces isomorphisms in homotopy groups through the same range. Since $BGL(\mathbb{C}[z]) \simeq BGL(\mathbb{C})$ the result follows.

□

6. The direct limit of the compactified spaces

The inclusion

$$V^d_{n,n+k}(\mathbb{C}[z]) \to V^{d+1}_{n,n+k}(\mathbb{C}[z])$$

induces an inclusion

$$V^d_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z]) \to V^{d+1}_{n,n+k}(\mathbb{C}[z])/GL_n(\mathbb{C}[z]).$$

These inclusion maps determine the following direct system.

$$\overline{\mathrm{Hol}}^A_1(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \overline{\mathrm{Hol}}^A_2(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \cdots$$

The results of the previous section show that the homotopy direct limit of this system is homotopy equivalent to $BGL_n(\mathbb{C})$. We now give a relatively quick self-contained proof that the ordinary direct limit is homotopy equivalent to $BGL_n(\mathbb{C})$.

Lemma 17. *The direct limit of*

$$V^1_{n,n+k}(\mathbb{C}[z]) \to V^2_{n,n+k}(\mathbb{C}[z]) \to V^3_{n,n+k}(\mathbb{C}[z]) \to \cdots$$

is contractible.

Proof:

It suffices to show that the homotopy groups of the direct limit are all zero. Pick any $d \in \mathbb{N}$ and let

$$i_d : V^d_{n,n+k}(\mathbb{C}[z]) \to V_{n,n+k}(\mathbb{C}[z])$$

be the inclusion map. The map

$$\phi_d : V^d_{n,n+k}(\mathbb{C}[z]) \times [0,1] \to V_{n,n+k}(\mathbb{C}[z])$$

defined by

$$\phi_d(M,t) = t[0_{n\times k} \quad z^{d+1} I_{n\times n}] + (1-t)M$$

where $0_{n\times k}$ is the $n\times k$ zero matrix and $I_{n\times n}$ is the $n\times n$ identity matrix is a homotopy from i_d to a constant map. Therefore the homotopy groups of the direct limit are all zero.

□

Theorem 18.

$$\lim_{\overrightarrow{d}} \overline{Hol}^A_d(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \simeq BGL_n(\mathbb{C})$$

Proof:

The group action of $GL_n(\mathbb{C}[z])$ on $V^d_{n,n+k}(\mathbb{C}[z])$ commutes with the inclusion maps. Therefore the quotient of the direct limit of

$$V^1_{n,n+k}(\mathbb{C}[z]) \to V^2_{n,n+k}(\mathbb{C}[z]) \to V^3_{n,n+k}(\mathbb{C}[z]) \to \cdots$$

is the direct limit of the quotient spaces

$$\overline{\mathrm{Hol}}_1^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \overline{\mathrm{Hol}}_2^A(\mathbb{C}P^1, G_{n,n+k}(\mathbb{C})) \to \cdots$$

Since $GL_n(\mathbb{C}[z])$ acts freely on the contractible space $V_{n,n+k}(\mathbb{C}[z])$, the quotient space of this direct limit is homotopy equivalent to $BGL_n(\mathbb{C}[z])$ and hence to $BGL_n(\mathbb{C})$ by Lemma 15.

□

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